## A PROOFS FOR OPTIMAL POLICY

Before presenting the main proofs, we present the following lemma as we use it in the proofs of the main lemmas and propositions later in this section.

LEMMA A.1. *Let $\{x_1, x_2, ..., x_m\}$ and $\{y_1, y_2, ..., y_n\}$ be two sets of real numbers where $n, m \in \mathbb{N}$. If $\forall\, y_j$, $\exists\, x_i$ such that $x_i \leq y_j$, then* $\min\{x_1, x_2, ..., x_m\} \leq \min\{y_1, y_2, ..., y_n\}$.

**Proof of Lemma 4.1:** We first prove Lemma 4.1.a and then Lemma 4.1.b. The proofs for Lemmas 4.1.c and 4.1.d are presented in the main body of the paper.

**Proof for Lemma 4.1.a ($T_a\tau \in \Theta$):** We define the auxiliary function $w_a$ for any $\tau \in \Theta$ as follows:

$$w_a(u, \mathbf{s}) = \begin{cases} \tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) & \text{if } u = 0 \\ \tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) & \text{if } u = 1 \end{cases}.$$

By definition:

$$T_a\tau(\mathbf{s}) = \min_{u \in \{0,1\}} w_a(u, \mathbf{s}), \tag{7}$$

where $u = 0$ corresponds to the decision to route the current job to the server under interference, and $u = 1$ corresponds to the decision to queue the current job.

The first and second differences of $w_a$ are:

$$D_{12}w_a(u, \mathbf{s}) = w_a(u, \mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) - w_a(u, \mathbf{s}) = \begin{cases} D_{12}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) & \text{if } u = 0 \\ D_{12}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) & \text{if } u = 1 \end{cases}, \tag{8}$$

$$D_{14}w_a(u, \mathbf{s}) = w_a(u, \mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) - w_a(u, \mathbf{s}) = \begin{cases} D_{14}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) & \text{if } u = 0 \\ D_{14}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) & \text{if } u = 1 \end{cases}, \tag{9}$$

$$D_{12}D_{14}w_a(u, \mathbf{s}) = D_{14}D_{12}w_a(u, \mathbf{s}) = \begin{cases} D_{12}D_{14}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) & \text{if } u = 0 \\ D_{12}D_{14}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) & \text{if } u = 1 \end{cases}, \tag{10}$$

$$D^2_{12}w_a(u, \mathbf{s}) = D_{12}D_{12}w_a(u, \mathbf{s}) = \begin{cases} D^2_{12}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) & \text{if } u = 0 \\ D^2_{12}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) & \text{if } u = 1 \end{cases}, \tag{11}$$

$$D^2_{14}w_a(u, \mathbf{s}) = D_{14}D_{14}w_a(u, \mathbf{s}) = \begin{cases} D^2_{14}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) & \text{if } u = 0 \\ D^2_{14}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) & \text{if } u = 1 \end{cases}. \tag{12}$$

We first discuss that $w_a \in \Theta$; i.e., $w_a$ preserves **P1**-**P6**:

- $w_a$ is non-decreasing in $\mathbf{e_1}$ and $\mathbf{e_2}$: By (8) and **P1** of $\tau$, each term of $D_{12}w_a(u, \mathbf{s}) \geq 0$; so, $w$ is also non-decreasing in $\mathbf{e_1}$ and $\mathbf{e_2}$.
- $w_a$ is non-decreasing in $\mathbf{e_1}$ and $\mathbf{e_4}$: By (9) and **P2** of $\tau$, each term of $D_{14}w_a(u, \mathbf{s} \geq 0$; so, $w$ is non-decreasing in $\mathbf{e_1}$ and $\mathbf{e_4}$.
- Supermodularity of $w_a$: By (10) and **P2** of $\tau$, we can conclude that $w_a(u, \mathbf{s})$ is also supermodular.
- Diagonal Dominance of $w_a$: Since $\tau$ is diagonally dominant, each term of (11) is greater than the corresponding term in (10), thus $w_a(u, \mathbf{s})$ is also diagonally dominant.
- Convexity of $w_a$ in $\mathbf{e_1}$ and $\mathbf{e_2}$: Each term of (11) is non-negative due to $\tau$ being convex in $\mathbf{e_1}$ and $\mathbf{e_2}$; so, $D^2_{12}w_a(u, \mathbf{s}) \geq 0$.
- Convexity of $w_a$ in $\mathbf{e_1}$ and $\mathbf{e_4}$: Each term of (12) is non-negative due to $\tau$ being convex in $\mathbf{e_1}$ and $\mathbf{e_4}$; so, $D^2_{14}w_a(u, \mathbf{s}) \geq 0$.

Note that for any state $\mathbf{s}$, $D_{12}w_a(u, \mathbf{s})$ is non-decreasing in $u$:

$$D_{12}w_a(u_1, \mathbf{s}) \leq D_{12}w_a(u_2, \mathbf{s}), \text{ where } u_1 \leq u_2. \tag{13}$$

If $u_1 = u_2$ we have $D_{12}w_a(u_1, \mathbf{s}) = D_{12}w_a(u_2, \mathbf{s})$, so $D_{12}w_a(u_1, \mathbf{s}) - D_{12}w_a(u_2, \mathbf{s}) = 0$. If $u_1 < u_2$, then $u_1 = 0$ and $u_2 = 1$, and we have:

$$\begin{aligned} D_{12}w_a(u_1, \mathbf{s}) - D_{12}w_a(u_2, \mathbf{s}) &= w_a(u_1, \mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) - w_a(u_1, \mathbf{s}) \\ &- w_a(u_2, \mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) + w_a(u_2, \mathbf{s}) = w_a(0, \mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) \\ &- w_a(0, \mathbf{s}) - w_a(1, \mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) + w_a(1, \mathbf{s}) = D_{12}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) \\ &- D_{12}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) = D_{12}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) - D_{12}\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) \\ &+ D_{12}\tau(\mathbf{s}) - D_{12}\tau(\mathbf{s}) = D_{12}D_{14}\tau(\mathbf{s}) - D^2_{12}\tau(\mathbf{s}) \leq 0 \end{aligned}$$

, since $\tau$ is diagonally dominant.

We now use $w_a(u, \mathbf{s})$ and its properties to show that $T_a\tau(\mathbf{s}) \in \Theta$.

**Proof that $T_a\tau$ is non-decreasing in $\mathbf{e_1}$ and $\mathbf{e_2}$.** We need to show that for any $\mathbf{s}$, $D_{12}T_a\tau \geq 0$, or $T_a\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) \geq T_a\tau(\mathbf{s})$, where

$$T_a\tau(\mathbf{s}) = \min\{\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}), \tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2})\}, \text{ and}$$
$$T_a\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) = \min\{\tau(\mathbf{s} + 2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_4}), \tau(\mathbf{s} + 2\mathbf{e_1} + 2\mathbf{e_2})\}.$$

We have $\tau(\mathbf{s} + 2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_4}) \geq \tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4})$ and $\tau(\mathbf{s} + 2\mathbf{e_1} + 2\mathbf{e_2}) \geq \tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2})$ by **P1** of $\tau$; thus, $T_a\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) \geq T_a\tau(\mathbf{s})$ by Lemma A.1. □

**Proof that $T_a\tau$ is non-decreasing in $\mathbf{e_1}$ and $\mathbf{e_4}$.** We need to show that for any $\mathbf{s}$, $D_{14}T_a\tau \geq 0$, or $T_a\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) \geq T_a\tau(\mathbf{s})$, where

$$T_a\tau(\mathbf{s}) = \min\{\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}), \tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2})\}, \text{ and}$$
$$T_a\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) = \min\{\tau(\mathbf{s} + 2\mathbf{e_1} + 2\mathbf{e_4}), \tau(\mathbf{s} + 2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_4})\}.$$

***Case 1***: $a_2(\mathbf{s}) \geq 2$. We have $\tau(\mathbf{s} + 2\mathbf{e_1} + 2\mathbf{e_4}) \geq \tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4})$ and $\tau(\mathbf{s} + 2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_4}) \geq \tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2})$ by P2 of $\tau$; thus, $T_a\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) \geq T_a\tau(\mathbf{s})$ by Lemma A.1.

***Case 2***: $a_2(\mathbf{s}) = 1$. In this case, an incoming job has to be queued in state $\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}$; i.e., $T_a\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) = \tau(\mathbf{s} + 2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_4})$, which is greater than or equal to the two possible outcomes of $T_a\tau(\mathbf{s})$, because $\tau$ is non-decreasing in $\mathbf{e_1}$ and $\mathbf{e_2}$ and non-decreasing in $\mathbf{e_1}$ and $\mathbf{e_4}$. □

**Proof for supermodularity of $T_a\tau$.** We need to show that $D_{12}D_{14}T_a\tau \geq 0$, or

$$T_a\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_4}) + T_a\tau(\mathbf{s} + \mathbf{e_1} + \mathbf{e_2}) \leq T_a\tau(\mathbf{s} + 2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_4}) + T_a\tau(\mathbf{s}) \tag{14}$$

***Case 1***: $a_2(\mathbf{s}) \geq 2$. Let $u_1, u_2 \in \{0, 1\}$ be the minimizers of $w_a$ in state $\mathbf{s} + 2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_4}$ and $\mathbf{s}$, respectively; i.e.:

$$\begin{aligned} &T_a\tau(\mathbf{s} + 2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_4}) = w_a(u_1, \mathbf{s} + 2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_4}), \text{ and} \\ &T_a\tau(\mathbf{s}) = w_a(u_2, \mathbf{s}). \end{aligned} \tag{15}$$

We prove (14) separately for $u_1 \geq u_2$ and $u_1 < u_2$:

$\underline{u_1 \geq u_2}$.

$$
\begin{aligned}
&T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \leq w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad + w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (7)},\\
&= w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)+w_a(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&\quad + w_a(u_2,\mathbf{s})-w_a(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-w_a(u_2,\mathbf{s})\\
&= w_a(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_a(u_2,\mathbf{s})+D_{12}w_a(u_2,\mathbf{s})\\
&\quad - D_{12}w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4), \text{by (8)},\\
&\leq w_a(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_a(u_2,\mathbf{s})+D_{12}w_a(u_1,\mathbf{s})\\
&\quad - D_{12}w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&, \text{by (13)}, D_{12}w_a(u_1,\mathbf{s}) \geq D_{12}w_a(u_2,\mathbf{s}) \text{ since } u_1 \geq u_2,\\
&= w_a(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_a(u_2,\mathbf{s})-D_{14}D_{12}w_a(u_1,\mathbf{s}), \text{by (10)},\\
&\leq w_a(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_a(u_2,\mathbf{s}), \text{since } w_a \text{ is supermodular},\\
&= T_a\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+T_a\tau(\mathbf{s}), \text{by (15)}.
\end{aligned}
$$

$\underline{u_1 < u_2}$. In this case we have $u_1 = 0$ and $u_2 = 1$.

$$
\begin{aligned}
&T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \leq w_a(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad + w_a(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (7)},\\
&= \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) = 2\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&\leq D_{14}^2\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)+2\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&, \text{since } D_{14}^2\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \geq 0,\\
&= \tau(\mathbf{s}+3\mathbf{e}_1+\mathbf{e}_2+2\mathbf{e}_4)-2\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&\quad + \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)+2\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&, \text{by definition of } D_{14}^2\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2),\\
&= \tau(\mathbf{s}+3\mathbf{e}_1+\mathbf{e}_2+2\mathbf{e}_4)+\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&= w_a(0,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_a(1,\mathbf{s}), \text{ by the definition of } w_a,\\
&= T_a\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+T_a\tau(\mathbf{s}), \text{ by (15)}.
\end{aligned}
$$

***Case 2***: $a_2(\mathbf{s}) = 1$. In this case, an incoming job has to be queued in states $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4$ and $\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4$; therefore, we need to show:

$$
\begin{aligned}
\tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)&\\
\geq T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-T_a\tau(\mathbf{s}).& \qquad (16)
\end{aligned}
$$

We let $u_1, u_2 \in \{0,1\}$ be the minimizers of $w_a$ in states $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2$ and $\mathbf{s}$, respectively; i.e.,

$$
T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) = w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{and } T_a\tau(\mathbf{s}) = w_a(u_2,\mathbf{s}). \qquad (17)
$$

We have four cases:

$\underline{u_1 = u_2 = 0}$.

$$
\begin{aligned}
&T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-T_a\tau(\mathbf{s}) = w_a(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-w_a(0,\mathbf{s}), \text{by (17)},\\
&= \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4), \text{by definition of } w_a.\\
&\leq \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+D_{12}^2\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4), \text{by } \mathbf{P5} \text{ of } \tau,\\
&= \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad + D_{12}\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-D_{12}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&= \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad + \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&\quad - \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&= \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4).
\end{aligned}
$$

$\underline{u_1 = 0 \text{ and } u_2 = 1}$.

$$
\begin{aligned}
&T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-T_a\tau(\mathbf{s}) = w(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-w(1,\mathbf{s}), \text{by (17)},\\
&= \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by definition of } w_a,\\
&= D_{14}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&\leq D_{14}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)+D_{14}D_{12}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by } \mathbf{P3} \text{ of } \tau,\\
&= D_{14}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)+D_{12}\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-D_{12}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&\leq D_{12}\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&, \text{since } u_1 = 0 \; D_{14}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \leq D_{12}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2),\\
&= \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4).
\end{aligned}
$$

$\underline{u_1 = 1 \text{ and } u_2 = 0}$.

$$
\begin{aligned}
&T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-T_a\tau(\mathbf{s}) = w_a(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-w_a(0,\mathbf{s}), \text{by (17)},\\
&= \tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)-\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4), \text{by definition of } w_a,\\
&= \tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)-\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+\tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)\\
&\quad - \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)+\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&\quad - \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) = \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&\quad + D_{14}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-D_{12}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&= \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+D_{14}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&\quad - D_{12}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+D_{14}\tau(\mathbf{s})-D_{14}\tau(\mathbf{s})+D_{12}\tau(\mathbf{s})-D_{12}\tau(\mathbf{s})\\
&= \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&\quad + D_{14}D_{12}\tau(\mathbf{s})-D_{12}D_{14}\tau(\mathbf{s})+D_{14}\tau(\mathbf{s})-D_{12}\tau(\mathbf{s})\\
&= \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+D_{14}\tau(s)-D_{12}\tau(s)\\
&, \text{since } D_{14}D_{12}\tau = D_{12}D_{14}\tau,\\
&\leq \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)-\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&, \text{since } u_2 = 0, D_{14}\tau(s) \leq D_{12}\tau(s).
\end{aligned}
$$

$\underline{u_1 = u_2 = 1}$.

$T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - T_a\tau(\mathbf{s}) = w_a(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - w_a(1,\mathbf{s})$, by (17),
$= \tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$, by definition of $w_a$,
$\leq \tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) + D^2_{14}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$, by **P6** of $\tau$,
$= \tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) + D_{14}\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)$
$- D_{14}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$
$= \tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) + \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4)$
$- \tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$
$= \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4) - \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)$
, by definition of $D_{14}\tau$ at $\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2$ and $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2$.

□

**Proof for diagonal dominance of $T_a\tau$.** We need to show that for any state $\mathbf{s}$, $D^2_{12}T_a\tau \geq D_{12}D_{14}T_a\tau$, or

$$T_a\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) + T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) \geq T_a\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2). \quad (18)$$

***Case 1***: $a_2(\mathbf{s}) \geq 2$. Let $u_1, u_2 \in \{0,1\}$ be the minimizers of $w_a$ at states $\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2$ and $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4$, respectively; i.e.:

$$T_a\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) = w_a(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2), \text{ and } T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) = w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4). \quad (19)$$

We prove (18) separately for $u_1 \geq u_2$ and $u_1 < u_2$:

$\underline{u_1 \geq u_2}$.

$T_a\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \leq w_a(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)$
$+ w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$, by (7),
$\leq w_a(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$
$+ D^2_{12}w_a(u_2,\mathbf{s}) - D_{12}D_{14}w_a(u_2,\mathbf{s})$, as $w_a$ is diagonally dominant,
$= w_a(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$
$+ D_{12}w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - D_{12}w_a(u_2,\mathbf{s}) - D_{12}w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)$
$+ D_{12}w_a(u_2,\mathbf{s}) = w_a(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$
$+ D_{12}w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - D_{12}w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)$
$\leq w_a(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$
$+ D_{12}w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - D_{12}w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)$
, by (13), $D_{12}w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \leq D_{12}w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$ as $u_2 \leq u_1$,
$= w_a(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$
$+ w_a(u_1,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - w_a(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$
$- w_a(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)$
$= w_a(u_1,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) + w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)$
$= T_a\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) + T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)$, by (19).

$\underline{u_1 < u_2}$. In this case we have $u_1 = 0$ and $u_2 = 1$.

$T_a\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \leq w_a(1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)$
$+ w_a(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$, by (7),
$= \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4) + \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)$, by definition of $w_a$,
$= w_a(0,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) + w_a(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)$, by definition of $w_a$,
$= T_a\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) + T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)$, by (19).

***Case 2***: $a_2(\mathbf{s}) = 1$. In this case, the only option in states $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4$ and $\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4$ is to queue; therefore, (21) simplifies to:

$$T_a\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \geq \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4) - \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4).$$

Let $u_1, u_2 \in \{0,1\}$ be the minimizers of $w_a$ at states $\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2$ and $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2$, respectively; i.e.,

$$T_a\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) = w_a(u_1,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) \text{ and } T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) = w_a(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2). \quad (20)$$

We have three feasible cases on the values of $u_1$ and $u_2$. The case where $u_1 = 1$ and $u_2 = 0$ corresponds to the decisions to utilize the under-interference server in state $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2$ and to queue in state $\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2$. However, if the server under interference is utilized in state $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2$, the system never transitions into $\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2$ from $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2$.

$\underline{u_1 = u_2 = 0}$.

$\tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4) - \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)$
$= w_a(0,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - w_a(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$, by definition of $w_a$,
$= T_a\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$, by (20).

$\underline{u_1 = 0 \text{ and } u_2 = 1}$.

$\tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4) - \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)$
$\leq \tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4) - \tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)$
, since $u_2 = 1$, $\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) \leq \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)$.,
$= w_a(0,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - w_a(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$, by definition of $w$.
$= T_a\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$, by (20).

$\underline{u_1 = u_2 = 1}$.

$\tau(\mathbf{s}+3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_4) - \tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) = D_{12}\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)$
$= D_{12}\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + D_{12}\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)$
$- D_{12}\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) + D_{12}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - D_{12}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$
$= D_{12}\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) + D_{14}D_{12}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - D^2_{12}\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$
$\leq D_{12}\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)$, by **P4** of $\tau$,
$= \tau(\mathbf{s}+3\mathbf{e}_1+3\mathbf{e}_2) - \tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)$
, by definition of $D_{12}\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)$,
$= w_a(1,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - w_a(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$, by definition of $w_a$,
$= T_a\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) - T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$, by (20).

□

**Proof for convexity of $T_a\tau(\mathbf{s})$ in $\mathbf{e}_1$ and $\mathbf{e}_2$.** Since $T_a\tau(\mathbf{s})$ is supermodular and diagonally dominant, it is also convex in $\mathbf{e}_1$ and $\mathbf{e}_2$. □

**Proof for convexity of $T_a\tau(\mathbf{s})$ in $\mathbf{e}_1$ and $\mathbf{e}_4$.** We need to show that $D^2_{14}T_a(\mathbf{s}) \geq 0$, or

$$T_a\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_4) + T_a\tau(\mathbf{s}) \geq 2T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4). \quad (21)$$

For convexity, there must be at least two available servers under interference.

***Case 1***: $a_2(\mathbf{s}) \geq 3$. Let $u_1, u_2 \in \{0, 1\}$ be the minimizers of $w_a$ in states $\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4$ and $\mathbf{s}$, respectively; i.e.:

$$T_a\tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) = w_a(u_1, \mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) \text{ and } T_a(\mathbf{s}) = w_a(u_2, \mathbf{s}). \tag{22}$$

$u_1 = u_2 = 1$.

$$\begin{aligned}
2T_a\tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) &\leq 2w_a(u_1, \mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) \\
&\leq 2w_a(u_1, \mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) + D_{14}^2 w_a(u_1, \mathbf{s}), \text{by } \mathbf{P5} \text{ of } w_a, \\
&= w_a(u_1, \mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + w_a(u_1, \mathbf{s}) \\
&= w_a(u_1, \mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + w_a(u_2, \mathbf{s}), \text{since } u_1 = u_2, \\
&= T_a\tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + T_a\tau(\mathbf{s}), \text{by (22)}.
\end{aligned}$$

$u_1 < u_2$. This case is not feasible. In this case $u_1 = 0$ and $u_2 = 1$; i.e, utilize in state $\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4$ and queue in state $\mathbf{s}$. The possible series of events that transition the process from $\mathbf{s}$ to $\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4$ are:

$\mathbf{s} \longrightarrow \mathbf{s} + \mathbf{e}_1 + \mathbf{e}_2$ (arrival and queue) $\longrightarrow$
$\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4$ (arrival and utilize) $\longrightarrow$
$\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4$ (service completion from a busy interference-free server) $\longrightarrow$
$\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4$ (arrival and utilize).

Queueing in $\mathbf{s}$ implies that there is no idle interference-free server in $\mathbf{s}$ ($a_3(\mathbf{s}) = 0$)—based on the non-idling policy for interference-free servers. Referring to the transitions above, a service completions from a busy interference-free server in $\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4$ (causing the transition to $\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4$) implies that there is one idle interference-free server in $\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4$. Based on the non-idling policy for interference-free servers, this implies that an arrival should be routed to the interference-free server in $\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4$, which is in contradiction with utilizing an under-interference server in this case.

$u_1 > u_2$.

$$\begin{aligned}
2T_a\tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) &\leq w_a(0, \mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) + w_a(1, \mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4), \text{by (7)}, \\
&= \tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + \tau(\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4), \\
&\leq \tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + \tau(\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4) + D_{12}D_{14}\tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) \\
&\quad, \text{by } \mathbf{P4} \text{ of } \tau, \\
&= \tau(\mathbf{s} + 3\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_4) + \tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) \\
&= w_a(1, \mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + w_a(0, \mathbf{s}), \text{by definition of } w_a, \\
&= T_a\tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + T_a\tau(\mathbf{s}), \text{by (22)}.
\end{aligned}$$

***Case 2***: $a_2(\mathbf{s}) = 2$. Let $u_1$ and $u_2$ be defined as before in case 1. Since there are only two available servers under interference in $\mathbf{s}$. The only option in $\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4$ is to queue. Thus $u_1 = 1$. So we have two cases on the value of $u_2$.

$u_2 = 0$.

$$\begin{aligned}
2T_a\tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) &\leq w_a(0, \mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) + w_a(1, \mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) \text{by (7)}. \\
&= \tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + \tau(\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4) \\
&\leq \tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + \tau(\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4) + D_{14}D_{12}\tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) \\
&\quad, \text{since } \tau \text{ is supermodular}, \\
&= \tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + \tau(\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4) + D_{12}\tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) \\
&\quad - D_{12}\tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) \\
&= \tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + \tau(\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4) + \tau(\mathbf{s} + 3\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_4) \\
&\quad - \tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) - \tau(\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4) + \tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) \\
&= \tau(\mathbf{s} + 3\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_4) + \tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) \\
&= w_a(1, \mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + w_a(0, \mathbf{s}), \text{by definition of } w_a, \\
&= T_a\tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + T_a\tau(\mathbf{s}), \text{by (22)}.
\end{aligned}$$

$u_2 = 1$.

$$\begin{aligned}
2T_a\tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4) &\leq 2w_a(1, \mathbf{s} + \mathbf{e}_1 + \mathbf{e}_4), \text{by (7)}, \\
&= 2\tau(\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4), \text{by definition of } w, \\
&\leq 2\tau(\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4) + D_{14}^2\tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_2) \\
&\quad, \text{since } \tau \text{ is convex in } \mathbf{e}_1 \text{ and } \mathbf{e}_4, \\
&= 2\tau(\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4) + \tau(\mathbf{s} + 3\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_4) \\
&\quad - 2\tau(\mathbf{s} + 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4) + \tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_2) \\
&= \tau(\mathbf{s} + 3\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_4) + \tau(\mathbf{s} + \mathbf{e}_1 + \mathbf{e}_2) \\
&= w_a(1, \mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + w_a(1, \mathbf{s}), \text{by definition of } w_a, \\
&= T_a\tau(\mathbf{s} + 2\mathbf{e}_1 + 2\mathbf{e}_4) + T_a\tau(\mathbf{s}), \text{by (22)}.
\end{aligned}$$

□

Since (21) holds in both cases $T_a\tau(\mathbf{s})$ is convex in $\mathbf{e}_1$ and $\mathbf{e}_4$. Thus $T_a\tau(\mathbf{s})$ has properties **P1**-**P6**, and is in $\Theta$. □

**Proof for $T_s\tau \in \Theta$:** We define the auxiliary function $w_s$ for any $\tau \in \Theta$ as follows:

$$w_s(u, \mathbf{s}) = \begin{cases} \tau(\mathbf{s} - \mathbf{e}_1 - \mathbf{e}_2) & \text{if } u = 0 \\ \tau(\mathbf{s} - \mathbf{e}_1 - \mathbf{e}_4) & \text{if } u = 1 \end{cases}.$$

Decisions after service completions matter when the departing job leaves the system with at least one idle under-interference server, and there is at least one job in the queue. By definition:

$$T_s\tau(\mathbf{s}) = \min_{u \in \{0,1\}} w_s(u, \mathbf{s}), \tag{23}$$

where $u = 0$ corresponds to the decision to rout the next job in the queue to the under-interference server that has just finished a service, and $u = 1$ corresponds to the decision to not utilize that server.

The first and second differences of $w_s$ are:

$$D_{12}w_s(u,\mathbf{s}) = w_s(u,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - w_s(u,\mathbf{s}) = \begin{cases} D_{12}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_2) & \text{if } u=0 \\ D_{12}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) & \text{if } u=1 \end{cases}, \tag{24}$$

$$D_{14}w_s(u,\mathbf{s}) = w_s(u,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) - w_s(u,\mathbf{s}) = \begin{cases} D_{14}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_2) & \text{if } u=0 \\ D_{14}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) & \text{if } u=1 \end{cases}, \tag{25}$$

$$D_{12}D_{14}w_s(u,\mathbf{s}) = D_{14}D_{12}w_s(u,\mathbf{s}) = \begin{cases} D_{12}D_{14}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_2) & \text{if } u=0 \\ D_{12}D_{14}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) & \text{if } u=1 \end{cases}, \tag{26}$$

$$D_{12}^2w_s(u,\mathbf{s}) = D_{12}D_{12}w_s(u,\mathbf{s}) = \begin{cases} D_{12}^2\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_2) & \text{if } u=0 \\ D_{12}^2\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) & \text{if } u=1 \end{cases}, \tag{27}$$

$$D_{14}^2w_s(u,\mathbf{s}) = D_{14}D_{14}w_s(u,\mathbf{s}) = \begin{cases} D_{14}^2\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_2) & \text{if } u=0 \\ D_{14}^2\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) & \text{if } u=1 \end{cases}. \tag{28}$$

We first discuss that $w_s \in \Theta$; i.e., **P1**-**P6** hold for $w_s$:

- $w_s$ is non-decreasing in $\mathbf{e}_1$ and $\mathbf{e}_2$: By (24) and **P1** of $\tau$, we have $D_{12}w_s(u,\mathbf{s}) \geq 0$ for each value of $u$; therefore, $w_s$ is non-decreasing in $\mathbf{e}_1$ and $\mathbf{e}_2$.
- $w_s$ is non-decreasing in $\mathbf{e}_1$ and $\mathbf{e}_4$: By (25) and **P2** of $\tau$, we have $D_{12}w_s(u,\mathbf{s}) \geq 0$ for each value of $u$; therefore, $w_s$ is non-decreasing in $\mathbf{e}_1$ and $\mathbf{e}_4$.
- Supermodularity of $w_s$: By (26) and **P2** of $\tau$, we have $D_{12}D_{14}w_s(u,\mathbf{s}) \geq 0$; therefore, $w_s$ is supermodular.
- Diagonal Dominance of $w_s$: Since $\tau$ is diagonally dominant we can see that each term of (27) is greater than or equal to the corresponding term in (26); therefore, $w_s$ is diagonally dominant under this new definition.
- Convexity of $w_s$ in $\mathbf{e}_1$ and $\mathbf{e}_2$: By (27) and **P4** of $\tau$, each term of $D_{12}^2w_s(u,\mathbf{s})$ is non-negative; therefore, $w_s$ is convex in $\mathbf{e}_1$ and $\mathbf{e}_2$.
- Convexity of $w_s$ in $\mathbf{e}_1$ and $\mathbf{e}_4$: By (28) and **P5** of $\tau$, we have $D_{14}^2w_s(u,\mathbf{s}) \geq 0$; therefore, $w_s$ is convex in $\mathbf{e}_1$ and $\mathbf{e}_4$.

Note that $D_{12}w_s(u,\mathbf{s})$ is non-decreasing in $u$:

$$D_{12}w_s(u_1,\mathbf{s}) \leq D_{12}w_s(u_2,\mathbf{s}) \text{ where } u_1 \leq u_2. \tag{29}$$

If $u_1 = u_2$, we have $D_{12}w_s(u1,\mathbf{s}) = D_{12}w_s(u_2,\mathbf{s})$. If $u_1 < u_2$ then $u_1 = 0$ and $u_2 = 1$, and we have:

$$\begin{aligned} D_{12}w_s(u_1,\mathbf{s}) - D_{12}w_s(u_2,\mathbf{s}) &= D_{12}w_s(0,\mathbf{s}) - D_{12}w_s(1,\mathbf{s}) \\ &= w_s(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - w_s(0,\mathbf{s}) - w_s(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) + w_s(1,\mathbf{s}) \\ &= \tau(\mathbf{s}) - \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_2) - \tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4) + \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \\ &= D_{12}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_2) - D_{12}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \\ &\quad + D_{12}\tau(\mathbf{s}-2\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_4) - D_{12}\tau(\mathbf{s}-2\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_4) \\ &= D_{14}D_{12}\tau(\mathbf{s}-2\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_4) - D_{12}^2\tau(\mathbf{s}-2\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_4) \\ &< 0, \text{since } \tau \text{ is diagonally dominant.} \end{aligned}$$

**Proof that $T_s\tau$ is non-decreasing in $\mathbf{e}_1$ and $\mathbf{e}_2$:** We need to show that for any $\mathbf{s}$, $D_{12}T_s\tau(\mathbf{s}) \geq 0$, or,

$$T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - T_s\tau(\mathbf{s}) \geq 0 \tag{30}$$

***Case 1***: $s_2(\mathbf{s}) \geq 1$. In this case, there will be at least two jobs in the queue in state $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2$; so a decision can be made in both states $\mathbf{s}$ and $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2$, and

$$\begin{aligned} T_s\tau(\mathbf{s}) &= \min\{\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_2), \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)\} \\ T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) &= \min\{\tau(\mathbf{s}), \tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4)\}. \end{aligned}$$

By **P1** of $\tau$, we have $\tau(\mathbf{s}) \geq \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_2)$ and $\tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4) \geq \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)$; therefore, $T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \geq T_s\tau(\mathbf{s})$ by Lemma A.1.

***Case 2***: $s_2(\mathbf{s}) = 0$. In this case there is no job to route to the new idle under-interference server in $\mathbf{s}$; so, (30) simplifies to:

$$T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \geq 0. \tag{31}$$

Let $u$ be the minimizer of $T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)$. If $u=0$ we have

$$\begin{aligned} T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - T_a\tau(\mathbf{s}) &= \tau(\mathbf{s}) - \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \\ &= D_{14}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \geq 0, \end{aligned}$$

by **P2** of $\tau$. If $u=1$, we have

$$\begin{aligned} T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) - T_a\tau(\mathbf{s}) &= \tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4) - \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \\ &= D_{12}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \geq 0, \end{aligned}$$

by **P1** of $\tau$. □

**Proof that $T_s\tau$ is non-decreasing in $\mathbf{e}_1$ and $\mathbf{e}_4$:** We need to show that for any $\mathbf{s}$, $D_{14}T_s\tau(\mathbf{s}) \geq 0$, or,

$$T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) - T_s\tau(\mathbf{s}) \geq 0, \tag{32}$$

***Case 1***: $s_2(\mathbf{s}) \geq 1$. In this case a decision can be made in both states $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4$ and $\mathbf{s}$, and

$$\begin{aligned} T_s\tau(\mathbf{s}) &= \min\{\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_2), \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)\} \\ T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) &= \min\{\tau(\mathbf{s}-\mathbf{e}_2+\mathbf{e}_4), \tau(\mathbf{s})\}, \end{aligned}$$

By **P2** of $\tau$, we have $\tau(\mathbf{s}-\mathbf{e}_2+\mathbf{e}_4) \geq \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_2)$ and $\tau(\mathbf{s}) \geq \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)$; therefore, $T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) \geq T_s\tau(\mathbf{s})$ by Lemma A.1.

***Case 2***: $s_2(\mathbf{s})$=0. In this case there will be no jobs in the queue in state $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4$; so, the new idle under-interference server remains idle as there is no job to route to the server. So, we have:

$$\begin{aligned} T_a\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) - T_a\tau(\mathbf{s}) &= \tau(\mathbf{s}) - \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \\ &= D_{14}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \geq 0, \end{aligned}$$

by **P2** of $\tau$. □

**Proof for Supermodularity of $T_s\tau$:** We need to show that $D_{12}D_{14}T_s\tau \geq 0$, or

$$T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) + T_s\tau(\mathbf{s}) \geq T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) + T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2). \tag{33}$$

***Case 1***: $s_2(\mathbf{s}) \geq 1$. In this case, there will be at least one jobs in the queue in $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4$ and at least two jobs in the queue in $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2$ and $\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4$. Thus a decision can be made in each state.

Let $u_1, u_2 \in \{0,1\}$ be the minimizers of $w_s$ in $\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4$ and $\mathbf{s}$, respectively; i.e.:

$$\begin{aligned} T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) &= w_s(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4), \text{ and} \\ T_s\tau(\mathbf{s}) &= w_s(u_2,\mathbf{s}). \end{aligned} \tag{34}$$

We prove (33) separately for $u_1 \geq u_2$ and $u_1 < u_2$:

$\underline{u_1 \geq u_2}$.

$$\begin{aligned}
&T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \leq w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad + w_s(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (23)},\\
&\quad = w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+w_s(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&\quad + w_s(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(u_2,\mathbf{s})\\
&\quad - w_s(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-w_s(u_2,\mathbf{s})\\
&\quad = w_s(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(u_2,\mathbf{s})+D_{12}w_s(u_2,\mathbf{s})\\
&\quad - D_{12}w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) \leq w_s(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&\quad + w_s(u_2,\mathbf{s})+D_{12}w_s(u_1,\mathbf{s})-D_{12}w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad, \text{by (29) and } u_1 \geq u_2,\\
&\quad = w_s(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(u_2,\mathbf{s})-D_{14}D_{12}w_s(u_1,\mathbf{s})\\
&\quad \leq w_s(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(u_2,\mathbf{s}), \text{since } w_s \text{ is supermodular},\\
&\quad = T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+T_s\tau(\mathbf{s}), \text{by (34)}.
\end{aligned}$$

$\underline{u_1 < u_2}$. In this case $u_1 = 0$ and $u_2 = 1$.

$$\begin{aligned}
&T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \leq w_s(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad + w_s(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (23)},\\
&\quad = \tau(\mathbf{s})+\tau(\mathbf{s}) \leq D_{14}^2\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)+2\tau(\mathbf{s}), \text{by } \mathbf{P2},\\
&\quad = \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)-2\tau(\mathbf{s})+\tau(\mathbf{s}-\mathbf{e}_1+\mathbf{e}_4)+2\tau(\mathbf{s})\\
&\quad = \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) = w_s(0,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)\\
&\quad + w_s(1,\mathbf{s})\\
&\quad = T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+T_s\tau(\mathbf{s}), \text{by (34)}.
\end{aligned}$$

***Case 2***: $a_2(\mathbf{s}) = 0$. In this case the queue will also be empty in $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4$. Therefore, we leave the server idle in these states after a service completion, i.e.,

$$T_s\tau(\mathbf{s}) = \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4), \text{ and } T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) = \tau(\mathbf{s}),$$

and (33) simplifies to:

$$T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) \geq \tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \tag{35}$$

We let $u_1, u_2 \in \{0,1\}$ be the minimizers of $w_s$ in $\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4$ and $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2$, respectively; i.e.:

$$\begin{aligned}
T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4) &= w_s(u1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4), \text{ and}\\
T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2) &= w_s(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2).
\end{aligned} \tag{36}$$

We consider four cases on the values of $u_1$ and $u_2$.

$\underline{u_1 = u_2 = 0}$.

$$\begin{aligned}
&\tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \leq \tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)+D_{14}^2\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)\\
&\quad, \text{by } \mathbf{P6} \text{ of } \tau,\\
&\quad = \tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4+\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)-2\tau(\mathbf{s})+\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad = \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)-\tau(\mathbf{s})\\
&\quad = w_s(0,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-w_s(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by definition of } w_s,\\
&\quad = T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (36)}.
\end{aligned}$$

$\underline{u_1 = 0 \text{ and } u_2 = 1}$.

$$\begin{aligned}
&\tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \leq \tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)+D_{14}^2\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)\\
&\quad, \text{by } \mathbf{P6} \text{ of } \tau,\\
&\quad = \tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)+\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)-2\tau(\mathbf{s})+\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad = \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)-\tau(\mathbf{s})\\
&\quad \leq \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)-\tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4), \text{since } u_2 = 1, \tau(\mathbf{s}) \geq \tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4),\\
&\quad = w_s(0,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-w_s(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by definition of } w_s,\\
&\quad = T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (36)}.
\end{aligned}$$

$\underline{u_1 = 1 \text{ and } u_2 = 0}$.

$$\begin{aligned}
&\tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \leq \tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)+D_{14}D_{12}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)\\
&\quad, \text{by } \mathbf{P3} \text{ of } \tau,\\
&\quad = \tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)+\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-\tau(\mathbf{s})-\tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4)\\
&\quad + \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) = \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-\tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4)\\
&\quad \leq \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-\tau(\mathbf{s}), \text{since } u_2 = 0, \tau(\mathbf{s}) \leq \tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4),\\
&\quad = w_s(1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-w_s(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by definition of } w_s,\\
&\quad = T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (36)}.
\end{aligned}$$

$\underline{u_1 = u_2 = 1}$.

$$\begin{aligned}
&\tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) \leq \tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)+D_{14}D_{12}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)\\
&\quad, \text{by } \mathbf{P3} \text{ of } \tau,\\
&\quad = \tau(\mathbf{s})-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)+\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-\tau(\mathbf{s})-\tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4)\\
&\quad + \tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4) = \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-\tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4)\\
&\quad = w_s(1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-w_s(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by definition of } w_s,\\
&\quad = T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)-T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (36)}.
\end{aligned}$$

□

**Proof for diagonal dominance of $T_s\tau$:** We need to show that $D_{12}^2T_s\tau(\mathbf{s}) \geq D_{12}D_{14}T_s\tau(\mathbf{s})$, or

$$\begin{aligned}
&T_s\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)+T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) \geq\\
&\qquad T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2).
\end{aligned} \tag{37}$$

***Case 1***: $a_2(\mathbf{s}) \geq 1$. In this case there will be at least one job in the queue in $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4$, at least two in $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2$ and $\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4$, and at least three in $\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2$. Therefore, utilizing/not utilizing decisions need to be made in each of these states.

Let $u_1, u_2 \in \{0,1\}$ be the minimizers of $w_s$ in $\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2$ and $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4$, respectively; i.e.:

$$\begin{aligned}
T_s\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2) &= w_s(u_1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2), \text{ and}\\
T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4) &= w_s(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4).
\end{aligned} \tag{38}$$

We prove (37) separately for $u_1 \geq u_2$ and $u_1 < u_2$:

$\underline{u_1 \geq u_2}$.

$$
\begin{aligned}
&T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\leq\\
&\quad w_s(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (23)},\\
&\quad \leq w_s(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&\quad +D_{12}^2w_s(u_2,\mathbf{s})-D_{12}D_{14}w_s(u_2,\mathbf{s}), \text{since } w \text{ is diagonally dominant},\\
&\quad = w_s(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&\quad +D_{12}w_s(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-D_{12}w_s(u_2,\mathbf{s})-D_{12}w_s(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad +D_{12}w_s(u_2,\mathbf{s})=w_s(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&\quad +D_{12}w_s(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-D_{12}w_s(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad \leq w_s(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&\quad +D_{12}w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)-D_{12}w_s(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad , \text{by (29) and } u_1\geq u_2,\\
&\quad = w_s(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&\quad +w_s(u_1,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)-w_s(u_1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&\quad -w_s(u_2,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad = w_s(u_1,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)+w_s(u_2,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad = T_s\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)+T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4), \text{by (38)}.
\end{aligned}
$$

$\underline{u_1 < u_2}$. In this case $u_1 = 0$ and $u_2 = 1$.

$$
\begin{aligned}
&T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\leq\\
&\quad w_s(1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (23)},\\
&\quad = \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)+\tau(\mathbf{s})=w_s(0,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)+w_s(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&\quad = T_s\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)+T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4), \text{by (38)}.
\end{aligned}
$$

***Case 2***: $a_2(\mathbf{s}) = 0$. In this case there is no job in queue in $\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4$ to utilize an idle under-interference server. Therefore (33) simplifies to:

$$
T_s\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)+\tau(\mathbf{s})\geq T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2). \tag{39}
$$

Let $u_1$ be defined as before in case 1:

$\underline{u_1 = 0}$.

$$
\begin{aligned}
&T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\leq\\
&\quad w_s(1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (23)},\\
&\quad = \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)+\tau(\mathbf{s}), \text{by definition of } w_s,\\
&\quad w_s(0,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)+\tau(\mathbf{s}), \text{by definition of } w_s,\\
&\quad = T_s\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)+\tau(\mathbf{s}), \text{by (38)}.
\end{aligned}
$$

$\underline{u_1 = 1}$.

$$
\begin{aligned}
&T_s\tau(\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\leq\\
&\quad w_s(1,\mathbf{s}+2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4)+w_s(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2), \text{by (23)},\\
&\quad = \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)+\tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4), \text{by definition of } w_s,\\
&\quad \leq \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)+\tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4)\\
&\quad +D_{12}^2\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)-D_{14}D_{12}\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4), \text{by } \textbf{P4} \text{ of } \tau,\\
&\quad = \tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)+\tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4)+\tau(\mathbf{s}+\mathbf{e}_1+2\mathbf{e}_2-\mathbf{e}_4)\\
&\quad -2\tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4)+\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)-\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_2)\\
&\quad +\tau(\mathbf{s})+\tau(\mathbf{s}+\mathbf{e}_2-\mathbf{e}_4)-\tau(\mathbf{s}-\mathbf{e}_1-\mathbf{e}_4)\\
&\quad = \tau(\mathbf{s}+\mathbf{e}_1+2\mathbf{e}_2-\mathbf{e}_4)+\tau(\mathbf{s})\\
&\quad = w_s(1,\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)+\tau(\mathbf{s}), \text{by definition of } w_s,\\
&\quad = T_s\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_2)+\tau(\mathbf{s}), \text{by (38)}
\end{aligned}
$$

□

**Proof for convexity of $T_s\tau$ in $\mathbf{e}_1$ and $\mathbf{e}_2$:** Since $T_s\tau(\mathbf{s})$ is supermodular and diagonally dominant, that implies $T_s\tau(\mathbf{s})$ is convex in $\mathbf{e}_1$ and $\mathbf{e}_2$. □

**Proof for convexity of $T_s\tau(\mathbf{s})$ in $\mathbf{e}_1$ and $\mathbf{e}_4$:** We need to show that $D_{14}^2T_s\tau(\mathbf{s})\geq 0$, or,

$$
2T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\leq T_s\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_4)+T_s\tau(\mathbf{s}). \tag{40}
$$

To observer convexity, $\mathbf{s}$ must have at least two idle under-interference servers. Also for $T_s\tau$ to exist there must be one busy server under interference so $k\geq 3$.

**When** $k = 2$: In order for $\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_4$ to be feasible, both servers need to be idle and under interference in $\mathbf{s}$, which means there is no decision to be made in $\mathbf{s}$, as no service completion could occur. Therefore, (40) simplifies to:

$$
2T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\leq T_s\tau(\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_4)+\tau(\mathbf{s}), \text{ or}
$$

depending on the conditions in $\mathbf{s}+2\mathbf{e}_1+2\mathbf{e}_4$:

$$
\begin{cases}
2T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\leq\tau(\mathbf{s}+\mathbf{e}_1-\mathbf{e}_2+2\mathbf{e}_4)+\tau(\mathbf{s})\\
\quad \text{, if queue is non-empty and interference server is utilized, and}\\
2T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\leq\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+\tau(\mathbf{s})\\
\quad \text{, if queue is empty or interference server is not utilized.}
\end{cases} \tag{41}
$$

We first prove the first inequality in (41):

$$
\begin{aligned}
2T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)&\leq w_s(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+w_s(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&=\tau(\mathbf{s}-\mathbf{e}_2+\mathbf{e}_4)+\tau(\mathbf{s})\\
&\leq\tau(\mathbf{s}+\mathbf{e}_1-\mathbf{e}_2+2\mathbf{e}_4)+\tau(\mathbf{s}), \text{by } \textbf{P2} \text{ of } \tau.
\end{aligned}
$$

Now, we prove the second inequality in (41):

$$
\begin{aligned}
2T_s\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)&\leq w_s(0,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+w_s(1,\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)\\
&=\tau(\mathbf{s}-\mathbf{e}_2+\mathbf{e}_4)+\tau(\mathbf{s})\\
&\leq\tau(\mathbf{s}+\mathbf{e}_1+\mathbf{e}_4)+\tau(\mathbf{s}), \text{by } \textbf{P1} \text{ of } \tau.
\end{aligned}
$$

Thus $T_s\tau(\mathbf{s})$ is convex in $\mathbf{e}_1$ and $\mathbf{e}_4$ when $k = 2$.

**When** $k\geq 3$: A decision can be made in all three states involved in (40).

*Case 1*: $a_2(\mathbf{s}) \geq 1$. Let $u_1, u_2 \in \{0,1\}$ be the minimizers of $w_s$ in $\mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}$ and $\mathbf{s}$, respectively; i.e.:

$$T_s\tau(\mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}) = w_s(u_1, \mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}) \text{ and } T_s\tau(\mathbf{s}) = w_s(u_2, \mathbf{s}). \quad (42)$$

We prove (40) separately for the cases where $u_1 = u_2$, $u_1 > u_2$, and $u_1 < u_2$:

$u_1 = u_2$.

$$
\begin{aligned}
2T_s\tau(\mathbf{s}+\mathbf{e_1}+\mathbf{e_4}) &\leq 2w_s(u_1, \mathbf{s}+\mathbf{e_1}+\mathbf{e_4})\\
&\leq 2w(u_1, \mathbf{s}+\mathbf{e_1}+\mathbf{e_4}) + D^2_{14}w(u_1, \mathbf{s}), \text{ since } w_s \text{ is convex in } \mathbf{e_1} \text{ and } \mathbf{e_4},\\
&= w_s(u_1, \mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}) + w_s(u_1, \mathbf{s})\\
&= w_s(u_1, \mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}) + w_s(u_2, \mathbf{s}), \text{ since } u_1 = u_2,\\
&= T_s\tau(\mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}) + T_s\tau(\mathbf{s}), \text{ by (42)}.
\end{aligned}
$$

$u_1 < u_2$. In this case $u_1 = 0$ and $u_2 = 1$. This corresponds to routing the next job to the idle server in $\mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}$ and leaving it idle in $\mathbf{s}$. This case is not feasible as it is not possible that the stochastic process visits $\mathbf{s}$ if the controller utilizes an idle under-interference server upon completion of a service from an under-interference server in $\mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}$.

$u_1 > u_2$. In this case $u_1 = 1$ and $u_2 = 0$. This is the case where we leave the server idle after a service completion in $\mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}$ and utilize it in state $\mathbf{s}$. Thus we have:

$$
\begin{aligned}
2T_s\tau(\mathbf{s}+\mathbf{e_1}+\mathbf{e_4}) &\leq w_s(0, \mathbf{s}+\mathbf{e_1}+\mathbf{e_4}) + w_s(1, \mathbf{s}+\mathbf{e_1}+\mathbf{e_4}), \text{ by (23)},\\
&= \tau(\mathbf{s}-\mathbf{e_2}+\mathbf{e_4}) + \tau(\mathbf{s}), \text{ by definition of } w_s,\\
&\leq \tau(\mathbf{s}-\mathbf{e_2}+\mathbf{e_4}) + \tau(\mathbf{s}) + D_{14}D_{12}\tau(\mathbf{s}-\mathbf{e_1}-\mathbf{e_2}), \text{ by } \mathbf{P4} \text{ of } \tau,\\
&= \tau(\mathbf{s}-\mathbf{e_2}+\mathbf{e_4}) + \tau(\mathbf{s}) + \tau(\mathbf{s}+\mathbf{e_1}+\mathbf{e_4}) - \tau(\mathbf{s}) - \tau(\mathbf{s}-\mathbf{e_2}+\mathbf{e_4})\\
&\quad + \tau(\mathbf{s}-\mathbf{e_1}-\mathbf{e_2}) = \tau(\mathbf{s}+\mathbf{e_1}+\mathbf{e_4}) + \tau(\mathbf{s}-\mathbf{e_1}-\mathbf{e_2})\\
&= w_s(1, \mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}) + w_s(0, \mathbf{s}), \text{ by definition of } w_s,\\
&= T_s\tau(\mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}) + T_s\tau(\mathbf{s}), \text{ by (42)}.
\end{aligned}
$$

*Case 2*: $a_2(\mathbf{s}) = 0$. In this case, the queue will also be empty in $\mathbf{s}+\mathbf{e_1}+\mathbf{e_4}$ and $\mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}$,and those state do not include any decision. Therefore, (40) simplifies to:

$$\tau(\mathbf{s}+\mathbf{e_1}+\mathbf{e_4}) + \tau(\mathbf{s}+\mathbf{e_1}-\mathbf{e_4}) \geq 2\tau(\mathbf{s}),$$

which becomes:

$$\tau(\mathbf{s}+\mathbf{e_1}+\mathbf{e_4}) - 2\tau(\mathbf{s} + \tau(\mathbf{s}+\mathbf{e_1}-\mathbf{e_4}) = D^2_{14}\tau(\mathbf{s}-\mathbf{e_1}-\mathbf{e_4} \geq 0,$$

by **P6** of $\tau$; Thus, (40) holds here as well. Since (40) holds, $T_s\tau(\mathbf{s})$ is convex in $\mathbf{e_1}$ and $\mathbf{e_4}$. □

$T_s\tau(\mathbf{s})$ satisfies properties **P1-P6**, which means it is also in $\Theta$. □

**Proof of Proposition 4.3.** We prove the proposition by first showing $i_2 \leq i_s$ and then showing $i_s \leq i_f$, both via contradiction.

$i_2 \leq i_s$: Let's assume that $i_2 > i_s$; this implies that $\exists k \in \mathbb{N}$ such that $i_s + k = i_2$. Suppose we are in state $\mathbf{s} = (i_2 - 1, n, 2, 0)$ prior to the next arrival. Since both servers are under interference and idle, the next decision happens at an arrival. If the optimal policy were followed the controller would queue the job and the system would be in state $\mathbf{s}+\mathbf{e_1}+\mathbf{e_2} = (i_2, n+1, 2, 0)$; if the controller followed a suboptimal action and routed the job to the idle server, the system would be in $\mathbf{s}+\mathbf{e_1}+\mathbf{e_4} = (i_2, n, 2, 1)$.

We define two processes $P_1$ and $P_2$, on the same probability space, such that they see the same events. Let $P_1$ and $P_2$ be initially in $\mathbf{s}+\mathbf{e_1}+\mathbf{e_2}$ and $\mathbf{s}+\mathbf{e_1}+\mathbf{e_4}$, respectively. Suppose the controller follows the optimal policy in $P_1$ and a suboptimal policy in $P_2$. We let $t$ be the first time either process has a decision epoch which would either be another arrival or a service completion to the server under interference in $P_2$. Up to now it is important to note that $P_1$ and $P_2$ have the same value.

***Case 1*:** The next decision epoch in an arrival. Since the optimal policy is followed in $P_1$, the controller would route that next customer to the idle server, and the system would enter $\mathbf{s}+2\mathbf{e_1}+\mathbf{e_2}+\mathbf{e_4}$; thus, the discounted expected total number of jobs in the system is $v(\mathbf{s}+2\mathbf{e_1}+\mathbf{e_2}+\mathbf{e_4})$. Suppose in $P_2$ the controller follows the optimal policy as well, then the system would enter $\mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}$, and the discounted expected total number of jobs in the system would be $v(\mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4})$. Comparing the two values we can see $v(\mathbf{s}+2\mathbf{e_1}+2\mathbf{e_4}) \leq v(\mathbf{s}+2\mathbf{e_1}+\mathbf{e_2}+\mathbf{e_4})$, since we are past the threshold value $i_2$.

***Case 2*:** The decision epoch is due to a service completion at the server under interference. In this case, $P_1$ would stay in $\mathbf{s}+\mathbf{e_1}+\mathbf{e_2}$ since that server is idle, and the discounted expected total number of jobs in the system would be $v(\mathbf{s}+\mathbf{e_1}+\mathbf{e_2})$. Suppose the controller did not route the next job to the server, then the system would transition to $\mathbf{s}$; if the controller followed the optimal policy from that point forward, the discounted expected number of jobs in the system would be $v(\mathbf{s})$. Comparing the two values we can see $v(\mathbf{s}) \leq v(\mathbf{s}+\mathbf{e_1}+\mathbf{e_2})$ since $v \in \Theta$ (by Corollary 2).

In both cases $P_2$ has a lower discounted expected total number of jobs in the system compared to $P_1$ after following a sub-optimal policy; since $P_2$'s initial state was reached via a sub-optimal action, this contradicts the optimality of the policy followed in $P_1$. Thus, $i_2 \leq i_s$.

$i_s \leq i_f$: Assume $i_s > i_f$. Then $\exists k \in \mathbb{N}$ such that $i_s = i_f + k$. Suppose we are in $\mathbf{s} = (i_s - 1, 1, 2, n)$ prior to the next arrival. When the arrival occurs, the system would transition to $\mathbf{s}+\mathbf{e_1}+\mathbf{e_2} = (i_s, 1, 2, n+1)$ if the optimal policy is followed and to $\mathbf{s}+\mathbf{e_1}+\mathbf{e_4} = (i_s, 2, 2, n)$ if a suboptimal action were taken. Let $P_1$, $P_2$, and $t$ be defined as in Case 1.

***Case 1*:** The next decision epoch is an arrival. In $P_1$ the controller would route the incoming job to the idle server in which case the system would transition to $\mathbf{s}+2\mathbf{e_1}+\mathbf{e_2}+\mathbf{e_4} = (i_s+1, 2, 2, n+1)$, and since the optimal policy is followed the discounted expected total number of jobs in the system would be $v(s+2e_1+e_2+e_4)$. In $P_2$, the only option for the controller is to queue the incoming job, so the system also transitions to $\mathbf{s}+2\mathbf{e_1}+\mathbf{e_2}+\mathbf{e_4}$. Thus, in the two processes the values are the same.

***Case 2*:** The next decision epoch is a service completion from one of the servers under interference. In $P_1$ after the service completion the controller would continue to utilize that under-interference server since both servers would be under interference and idle; so, the state transitions into $\mathbf{s} = (i_s - 1, 0, 2, n)$. Following the optimal policy the discounted expected total number of jobs in the system would be $v(\mathbf{s})$. Suppose the controller continued to utilize in $P_2$ as well; then, the system would transition into $\mathbf{s}-\mathbf{e_2}+\mathbf{e_4} = (i_s - 1, 1, 2, n-1)$. Comparing the two values we can see that

$v(\mathbf{s} - \mathbf{e}_2 + \mathbf{e}_4) \leq v(\mathbf{s})$; since both servers are idle and under interference in $s$, it is better to keep one of the servers busy than keeping both idle following the optimal policy.

Since in both cases we have shown the value from following the sub-optimal policy in $P_2$ results in a value that is less than or equal to that in $P_1$, it contradicts the optimality of the policy in $P_1$; thus, $i_s \leq i_f$. □

## B TABLE OF ADDITIONAL RESULTS

Table 4 presents additional numerical results showing the improvement of the optimal policy over the non-idling policy for a wide range of practical parameter settings in 2-VM systems.

For a 2-VM system, the optimal dispatching policy is described by a set of three threshold values which we present as $N_i$, $N_{ib}$ and $N_{ii}$. Tables 5, 6, and 6 present the threshold values $N_i$, $N_{ib}$, and $N_{ii}$, respectively, of the optimal dispatching policy for the same range of practical parameter settings given in Table 4 (2-VM systems). $N_i$ ($N_i$+2) is the minimum number of jobs in the system for which an arrival event (service completion event at the under-interference server) will cause the dispatcher to send a job to the idle under-interference server when the 2-VM system has a busy interference-free server and an idle under-interference server (a busy interference-free server and a busy under-interference server). $N_{ib}$ ($N_{ib}$+2) is the minimum number of jobs in the system for which an arrival event (service completion event at one of the under-interference server) will cause the dispatcher to send a job to the idle under-interference server when the 2-VM system has a busy interference-free server and an idle under-interference server (both servers busy and under-interference). Finally, $N_{ii}$ ($N_{ii}$+2) is the minimum number of jobs in the system for which an arrival event (service completion event at the under-interference server) will cause the dispatcher to send a job to one of the idle under-interference server when the 2-VM system has two idle under-interference server (an idle and busy under-interference server).

| | | $\alpha_L$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.02 | | | 0.10 | | | 1 | | | 10 | | | 50 | | |
| $\rho$ | $\mu_L/\mu_H$ | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | |
| | | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 |
| 0.10 | 0.01 | 2.78 | 1.60 | 0.53 | 10.41 | 6.82 | 2.29 | 22.58 | 21.94 | 7.32 | 10.83 | 15.76 | 21.36 | 2.23 | 3.91 | 8.90 |
| | 0.02 | 2.45 | 1.40 | 0.43 | 9.14 | 6.00 | 1.95 | 20.24 | 20.32 | 7.44 | 9.91 | 14.28 | 18.05 | 2.12 | 3.70 | 8.28 |
| | 0.10 | 8.84 | 10.54 | 6.97 | 9.57 | 11.31 | 7.05 | 9.88 | 12.59 | 7.34 | 5.24 | 7.19 | 4.52 | 1.34 | 2.21 | 3.84 |
| | 0.20 | 4.77 | 6.87 | 5.62 | 4.74 | 6.83 | 5.58 | 4.46 | 6.44 | 5.20 | 2.62 | 3.77 | 2.69 | 0.62 | 0.87 | 0.32 |
| | 0.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0.25 | 0.01 | 0.98 | 0.58 | 0.27 | 4.03 | 2.64 | 1.21 | 12.13 | 11.92 | 5.78 | 8.33 | 11.75 | 13.58 | 2.29 | 3.71 | 6.86 |
| | 0.02 | 0.94 | 0.56 | 0.24 | 3.88 | 2.55 | 1.13 | 11.95 | 11.95 | 6.07 | 7.89 | 11.02 | 11.49 | 2.17 | 3.50 | 6.29 |
| | 0.10 | 0.77 | 0.48 | 0.26 | 2.91 | 2.09 | 1.17 | 7.34 | 8.46 | 5.74 | 4.90 | 6.96 | 5.56 | 1.34 | 2.05 | 2.64 |
| | 0.20 | 1.12 | 1.15 | 4.54 | 2.53 | 2.91 | 4.64 | 3.45 | 4.85 | 5.08 | 2.23 | 3.33 | 3.12 | 0.61 | 0.87 | 0.34 |
| | 0.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0.50 | 0.01 | 0.13 | 0.11 | 0.09 | 0.57 | 0.53 | 0.40 | 2.38 | 2.78 | 2.31 | 2.74 | 4.03 | 5.24 | 1.16 | 1.88 | 3.45 |
| | 0.02 | 0.11 | 0.10 | 0.08 | 0.49 | 0.45 | 0.39 | 2.18 | 2.60 | 2.46 | 2.58 | 3.82 | 4.62 | 1.09 | 1.76 | 3.08 |
| | 0.10 | 0.08 | 0.06 | 0.05 | 0.34 | 0.27 | 0.25 | 1.31 | 1.53 | 1.79 | 1.55 | 2.37 | 3.00 | 0.61 | 0.96 | 1.20 |
| | 0.20 | 0.06 | 0.04 | 0.03 | 0.23 | 0.18 | 0.17 | 0.70 | 0.87 | 1.05 | 0.71 | 1.12 | 1.53 | 0.24 | 0.35 | 0.06 |
| | 0.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0.75 | 0.01 | 0.01 | 0.01 | 0.02 | 0.02 | 0.03 | 0.11 | 0.20 | 0.26 | 0.65 | 0.58 | 0.88 | 1.55 | 0.38 | 0.62 | 1.23 |
| | 0.02 | 0.00 | 0.01 | 0.02 | 0.02 | 0.03 | 0.09 | 0.19 | 0.25 | 0.64 | 0.54 | 0.83 | 1.42 | 0.35 | 0.57 | 1.09 |
| | 0.10 | 0.00 | 0.00 | 0.01 | 0.02 | 0.02 | 0.04 | 0.12 | 0.16 | 0.36 | 0.30 | 0.48 | 0.92 | 0.19 | 0.31 | 0.40 |
| | 0.20 | 0.00 | 0.00 | 0.00 | 0.01 | 0.01 | 0.02 | 0.07 | 0.09 | 0.19 | 0.13 | 0.21 | 0.41 | 0.03 | 0.05 | 0.00 |
| | 0.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0.90 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.02 | 0.04 | 0.05 | 0.17 | 0.14 | 0.22 | 0.46 | 0.12 | 0.19 | 0.40 |
| | 0.02 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.02 | 0.04 | 0.05 | 0.15 | 0.13 | 0.21 | 0.43 | 0.11 | 0.18 | 0.35 |
| | 0.10 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.02 | 0.03 | 0.09 | 0.08 | 0.12 | 0.26 | 0.06 | 0.09 | 0.13 |
| | 0.20 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.01 | 0.02 | 0.04 | 0.03 | 0.04 | 0.10 | 0.00 | 0.00 | 0.00 |
| | 0.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0.95 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.02 | 0.02 | 0.07 | 0.06 | 0.10 | 0.21 | 0.05 | 0.09 | 0.19 |
| | 0.02 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.01 | 0.02 | 0.07 | 0.06 | 0.09 | 0.19 | 0.05 | 0.08 | 0.17 |
| | 0.10 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.01 | 0.04 | 0.03 | 0.05 | 0.11 | 0.03 | 0.04 | 0.06 |
| | 0.20 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.02 | 0.01 | 0.02 | 0.04 | 0.00 | 0.00 | 0.00 |
| | 0.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

**Table 4: Percentage improvement for a 2-VM system with $\mu_H = 100$ and other parameters as defined in the table.**

| | | $\alpha_L$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.02 | | | 0.10 | | | 1 | | | 10 | | | 50 | | |
| $\rho$ | $\mu_L/\mu_H$ | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | |
| | | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 |
| 0.1 | 0.01 | 80 | 81 | 84 | 77 | 77 | 76 | 53 | 52 | 43 | 18 | 17 | 13 | 7 | 7 | 5 |
| | 0.02 | 40 | 41 | 41 | 39 | 39 | 38 | 32 | 31 | 27 | 14 | 13 | 10 | 6 | 6 | 4 |
| | 0.10 | 8 | 8 | 8 | 8 | 8 | 8 | 7 | 7 | 7 | 5 | 5 | 4 | 3 | 3 | 2 |
| | 0.20 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 3 | 3 | 3 | 2 | 2 | 2 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0.25 | 0.01 | 54 | 58 | 72 | 52 | 55 | 66 | 37 | 38 | 38 | 14 | 14 | 12 | 6 | 6 | 5 |
| | 0.02 | 27 | 29 | 36 | 27 | 28 | 34 | 22 | 23 | 24 | 10 | 10 | 9 | 5 | 5 | 4 |
| | 0.10 | 6 | 6 | 7 | 6 | 6 | 6 | 5 | 6 | 6 | 4 | 4 | 4 | 3 | 3 | 2 |
| | 0.20 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 2 | 2 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0.5 | 0.01 | 15 | 20 | 49 | 15 | 20 | 45 | 14 | 16 | 27 | 9 | 9 | 9 | 5 | 5 | 4 |
| | 0.02 | 10 | 12 | 24 | 10 | 11 | 23 | 9 | 11 | 17 | 7 | 7 | 7 | 4 | 4 | 4 |
| | 0.10 | 3 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 4 | 3 | 3 | 3 | 2 | 2 | 2 |
| | 0.20 | 2 | 2 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0.75 | 0.01 | 8 | 8 | 26 | 7 | 8 | 25 | 7 | 8 | 17 | 6 | 6 | 8 | 4 | 4 | 4 |
| | 0.02 | 6 | 6 | 14 | 6 | 6 | 13 | 6 | 6 | 11 | 5 | 5 | 6 | 4 | 4 | 3 |
| | 0.10 | 3 | 3 | 4 | 3 | 3 | 4 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 2 |
| | 0.20 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0.9 | 0.01 | 6 | 7 | 15 | 6 | 7 | 15 | 6 | 7 | 13 | 5 | 6 | 7 | 4 | 4 | 4 |
| | 0.02 | 5 | 5 | 9 | 5 | 5 | 9 | 5 | 5 | 9 | 4 | 5 | 5 | 3 | 3 | 3 |
| | 0.10 | 2 | 3 | 3 | 2 | 3 | 3 | 2 | 3 | 3 | 2 | 2 | 3 | 2 | 2 | 2 |
| | 0.20 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0.95 | 0.01 | 6 | 6 | 13 | 6 | 6 | 13 | 6 | 6 | 11 | 5 | 5 | 7 | 4 | 4 | 4 |
| | 0.02 | 5 | 5 | 8 | 5 | 5 | 8 | 5 | 5 | 8 | 4 | 4 | 5 | 3 | 3 | 3 |
| | 0.10 | 2 | 3 | 3 | 2 | 3 | 3 | 2 | 2 | 3 | 2 | 2 | 3 | 2 | 2 | 2 |
| | 0.20 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 5: Optimum threshold $N_i$ for a 2-VM system with $\mu_H = 100$ and other parameters as defined in the table.**

| | | $\alpha_L$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.02 | | | 0.10 | | | 1 | | | 10 | | | 50 | | |
| | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | |
| $\rho$ | $\mu_L/\mu_H$ | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 |
| 0.1 | 0.01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 11 | 11 | 8 | 5 | 5 | 4 |
| | 0.02 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 7 | 7 | 5 | 4 | 4 | 3 |
| | 0.10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 1 |
| | 0.20 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0.25 | 0.01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 8 | 8 | 7 | 5 | 5 | 4 |
| | 0.02 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 5 | 4 | 4 | 4 | 3 |
| | 0.10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0.20 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0.5 | 0.01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 4 | 4 | 5 | 4 | 4 | 3 |
| | 0.02 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 3 | 3 | 3 | 3 |
| | 0.10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0.20 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0.75 | 0.01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 4 | 3 | 3 | 3 |
| | 0.02 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| | 0.10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0.20 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0.9 | 0.01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 |
| | 0.02 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| | 0.10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0.20 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0.95 | 0.01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 2 | 3 | 3 |
| | 0.02 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| | 0.10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0.20 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0.50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 6: Optimum threshold $N_{ib}$ for a 2-VM system with $\mu_H = 100$ and other parameters as defined in the table.**

| | | $\alpha_L$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.02 | | | 0.10 | | | 1 | | | 10 | | | 50 | | |
| $\rho$ | $\mu_L/\mu_H$ | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | | $\alpha_H/\alpha_L$ | | |
| | | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 | 0.1 | 0.2 | 1 |
| 0.1 | 0.01 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 11 | 11 | 8 | 5 | 5 | 4 |
| | 0.02 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 | 7 | 5 | 4 | 4 | 3 |
| | 0.10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 1 |
| | 0.20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| | 0.50 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.25 | 0.01 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 8 | 7 | 4 | 4 | 4 |
| | 0.02 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 5 | 4 | 3 | 3 | 3 |
| | 0.10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| | 0.20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0.50 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.5 | 0.01 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 4 | 5 | 3 | 3 | 3 |
| | 0.02 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 3 | 2 | 2 | 2 |
| | 0.10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| | 0.20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0.50 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.75 | 0.01 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 3 | 3 | 3 | 3 |
| | 0.02 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 2 | 2 |
| | 0.10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| | 0.20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0.50 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.9 | 0.01 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 3 | 2 | 2 | 2 |
| | 0.02 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 2 |
| | 0.10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0.20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0.50 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.95 | 0.01 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 3 | 2 | 2 | 2 |
| | 0.02 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 2 |
| | 0.10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0.20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0.50 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 7: Optimum threshold $N_{ii}$ for a 2-VM system with $\mu_H = 100$ and other parameters as defined in the table.**