The Query Complexity of Bayesian Auctions*

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Abstract

Generating good revenue is one of the most important problems in Bayesian auction design, and many (approximately) optimal dominant-strategy incentive compatible (DSIC) Bayesian mechanisms have been constructed for various auction settings. However, most existing studies do not consider the complexity for the seller to carry out the mechanism. It is assumed that the seller knows "each single bit" of the distributions and is able to optimize perfectly based on the entire distributions. Unfortunately, this assumption is very strong and may not hold in reality. For example, when the value distributions have exponentially large supports or do not have succinct representations, it is unclear how to find the optimal allocation and prices in many existing mechanisms.

In this work we consider, for the first time, the *query complexity* of Bayesian mechanisms. We only allow the seller to have limited oracle accesses to the players' value distributions, via *quantile queries* and *value queries*. For a large class of auction settings, we prove *logarithmic* lower-bounds for the query complexity for any DSIC Bayesian mechanism to be a constant approximation to the optimal revenue. For single-item auctions, unit-demand auctions and additive auctions, we prove *tight* upper-bounds for the query complexity by constructing efficient DSIC Bayesian mechanisms. Finally, we show how to use our results to construct *sampling* mechanisms that use polynomially many samples from the distributions. Indeed, we provide the first *constructive* sampling mechanisms for unit-demand auctions and additive auctions.

Keywords: mechanism design, the complexity of Bayesian mechanisms, query complexity, quantile queries, value queries

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1 Introduction

An important problem in Bayesian mechanism design is to design auction mechanisms that (approximately) maximize the seller's expected revenue. More precisely, in a Bayesian multi-item auction a seller has m heterogenous items to sell to n players. Each player i has a private value for each item j, v_{ij}; and each v_{ij} is independently drawn from some prior distribution $\mathcal{D}_{ij}$. When the prior distribution $\mathcal{D} \triangleq \times_{ij}\mathcal{D}_{ij}$ is of *common knowledge* to both the seller and the players, optimal Bayesian incentive-compatible (BIC) mechanisms have been discovered for various auction settings [26, 12, 3, 4], where all players reporting their true values forms a Bayesian Nash equilibrium. When there is no common prior but the seller knows $\mathcal{D}$, many (approximately) optimal dominant-strategy incentive-compatible (DSIC) Bayesian mechanisms have been designed [26, 27, 8, 24, 30, 5], where it is each player's *dominant strategy* to report his true values.

However, the *complexity* for the seller to carry out such mechanisms is largely unconsidered in the literature. Most existing Bayesian mechanisms require that the seller has full access to the prior distribution $\mathcal{D}$ and is able to carry out all required optimizations based on $\mathcal{D}$, so as to compute the allocation and the prices of the auction. Unfortunately, the seller is not so knowledgeable or powerful in many real-world scenarios. For example, if the supports of the distributions are exponentially large (in m and n), or if the distributions are continuous and do not have succinct representations, it is hard for the seller to write out "each single bit" of the distributions or precisely carry out arbitrary optimizations based on them. In fact, even with a single player and a single item, when the value distribution is irregular, computing the optimal price in time that is much smaller than the size of the support is not an easy task. Thus, a natural and important question to ask is *how much the seller should know about the distributions in order to obtain approximately optimal revenue.*

In this work we consider, for the first time, the *query complexity* of Bayesian mechanisms. In particular, the seller can only access the distributions by making oracle queries. Two natural types of queries are allowed, *quantile queries* and *value queries*. That is, the seller queries the oracle with specific quantiles (respectively, values), and the oracle returns the corresponding values (respectively, quantiles) in the underlying distributions.

These two types of queries happen a lot in market study. Indeed, the seller may wish to know what is the price he should set so that half of the consumers would purchase his product; or if he sets the price to be 200 dollars, how many consumers would buy it. Another important scenario where such queries naturally come up is in databases. Indeed, although the seller may not know the distribution, some powerful institutes, say the Office for National Statistics, may actually have such information all figured out and stored in its database. As in most database applications, it may be neither necessary nor feasible for the seller to download the whole distribution to his local machines. Rather, he would like to access the distribution via queries to the database. Other types of queries are of course possible, and will be considered in future works.

In this work we focus on *non-adaptive* queries. That is, the seller makes all oracle queries simultaneously, before the auction starts. This is also natural in both database and market study scenarios, and adaptive queries will again be considered in future works.

A closely related area in the literature is sampling mechanisms [11, 23, 15, 25, 14, 17]. Here it is assumed that the seller does not know $\mathcal{D}$, but is able to observe independent samples from $\mathcal{D}$ before the auction begins. The *sample complexity* measures how many samples the seller needs so as to obtain a good approximation to the optimal Bayesian revenue. As will become clear in the technical part of this paper, in some sense, queries can be seen as *targeted samples*, where the seller actively asks the information he needs rather than passively learns about it from random samples. As such, it is intuitive that queries are more efficient than samples, but it is *a priori* unclear how efficient queries can be. Our main results answer this question quantitatively and show that query complexity can be exponentially smaller than sample complexity: the former is *logarithmic* in the "size" of the distributions, while the latter is polynomial. Indeed, *active queries are significantly more powerful than passive notifications.*

1.1 Main Results

We would like to understand both lower- and upper-bounds for the query complexity of approximately optimal Bayesian auctions. In this work, we mainly consider three widely studied settings: single-item auctions, unit-demand auctions, and additive auctions. Our main results, their implications on sample complexity, as well as the best-known sample complexity in the literature are summarized in Table 1. Note that we allow arbitrary unbounded distributions that satisfy *small-tail assumptions*, with formal definitions deferred to Section 5.1. Similar assumptions are widely adopted in sampling mechanisms [28, 14], to deal with irregular distributions with unbounded supports.

Auctions	Distributions	Query Complexity	Sample Complexity
Single-Item	Regular	$\Omega(n\epsilon^{-1})$, $O(n\epsilon^{-1}\log\frac{n}{\epsilon})$	$\Omega(\max\{n\epsilon^{-1},\epsilon^{-3}\})$ [11, 23], $\tilde{O}(n\epsilon^{-4})$ [14]
	Bounded in $[1, H]$	$\Theta(n\epsilon^{-1}\log H)$	$\Omega(H\epsilon^{-2})$ [23], $\tilde{O}(nH\epsilon^{-3})$ [14]
	Unbounded & Small Tail	$O(-n\epsilon^{-1}\log h(\frac{2\epsilon}{3(1+\epsilon)}))$	$\tilde{O}(h^{-2}(\frac{2\epsilon}{3(1+\epsilon)})\epsilon^{-2})$
Unit-Demand	Bounded in $[1, H]$	$\forall c>1$: $\Omega(\frac{mn\log H}{\log c})$ $\forall c>24$: $O(\frac{mn\log H}{\log(c/24)})$	$\forall c>24$: $\tilde{O}(n^2m^4H^2c^4(\frac{c}{24}-1)^{-4})$; $\forall c>27$, non-constructive: $\tilde{O}(nm^2H^2(\frac{c}{27}-1)^{-2})$[25]
	Unbounded & Small Tail	$\forall c>24$: $O(-\frac{n\log h(\frac{2c-48}{3c})}{\log(c/24)})$	$\forall c>24$: $\tilde{O}(h^{-2}(\frac{2c-48}{3c})c^2(\frac{c}{24}-1)^{-2})$
Additive	Bounded in $[1, H]$	$\forall c>1$: $\Omega(\frac{mn\log H}{\log c})$ $\forall c>8$: $O(\frac{mn\log H}{\log(c/8)})$	$\forall c>8$: $\tilde{O}(\frac{n^2m^4H^2c^2}{(c-8)^2}(\frac{1}{2}-\frac{1}{(\frac{c+32}{40})^{1/m}+1})^2)$
	Unbounded & Small Tail	$\forall c>8$: $O(-\frac{m^2n\log h(\frac{c-8}{10c})}{\log(c/8)})$	$\forall c>8$: $\tilde{O}(h^{-2}(\frac{c-8}{10c})(\frac{1}{2}-\frac{1}{(\frac{c+32}{40})^{1/m}+1})^2)$

Table 1: Our main results. Here $h(\cdot)<1$ is the tail function in the small-tail assumptions. For single-item auctions, the revenue is a $(1+\epsilon)$-approximation to the optimal BIC revenue, with ϵ sufficiently small. For unit-demand and additive auctions, the revenue is a c-approximation for some constant c. A sample contains a valuation profile of the players, that is, mn values; while a query contains a single value or a single quantile.

Note that our lower- and upper-bounds on query complexity are *tight* for bounded distributions. As will become clear in Section 3 and Appendix A, our lower-bounds allow the seller to make both value and quantile queries, and actually apply to many other multi-player multi-item auctions, where each player's valuation function is *succinct sub-additive*: formal definitions deferred to Appendix A. The lower-bounds also allow randomized queries and randomized mechanisms.

For the upper-bounds, all our queries are deterministic, and our mechanisms only make one type of queries: value queries for bounded distributions; and quantile queries in the other cases; see Sections 4 and 5. In the construction and the analysis of our mechanisms, we show how to discretize the value space and the quantile space of the prior distribution, while losing only a small fraction in revenue.

Finally, the design of query mechanisms facilitates the design of sampling mechanisms. Indeed, if the seller can observe enough samples from $\mathcal{D}$, then he can mimic quantile queries and apply query mechanisms; see Section 6 for more details. In particular, although [25] shows that it is possible to approximate the optimal revenue using polynomially many samples, we provide the first *constructive* sampling mechanisms for multi-parameter auctions, and our sampling mechanism for unit-demand auctions has a better approximation ratio than [25].

1.2 Future Directions

As this is the first time the query complexity of Bayesian auctions is considered, many interesting future directions are worth exploring.

First, as mentioned, we focus on non-adaptive queries in this work. One can imagine more powerful mechanisms using *adaptive* queries, where the seller's later queries depend on the oracle's responses to former ones. Allowing adaptive queries may further reduce the query complexity. It is intriguing to design approximately optimal Bayesian mechanisms with adaptive queries, or prove that even with such queries, the query complexity cannot be much better than our lower-bounds.

Another interesting direction is when the answers of the oracle contain noise. In this case, the distributions learnt by the seller may be within a small distance from the "true distributions" under the oracle's precise answers. Can one design *robust* mechanisms that maximize the minimum expected revenue across all possible true distributions?

Finally, if the players' value distributions are known by some experts, then the seller can use the experts as oracles. Indeed, we are able to design proper scoring rules [2, 7] for the seller to elicit truthful answers from the experts for his queries. If the experts are actually players in the auctions, then they have their own stakes about the final allocation and prices, and it would be interesting to see how the seller can still use them as oracles and get truthful answers for his queries, while keeping them truthful about their own values. See [10] for more discussions on this front.

1.3 Additional Related Works

The complexity of auctions is an important topic in the literature, and several complexity measures have been considered. Following the taxation principle [19, 18], [21] defines the *menu complexity* of truthful auctions. For a single additive buyer, [13] shows the optimal Bayesian auction for revenue can have an infinite menu size or a continuum of menu entries, and [1] shows a constant approximation under finite menu complexity. Recently, [16] considers the taxation, communication, query and menu complexities of truthful combinatorial auctions, and shows important connections among them. The queries considered there are totally different from ours: we are concerned with the complexity of accessing the players' value distributions in Bayesian settings, and [16] is concerned with the complexity of accessing the players' valuation functions in non-Bayesian settings.

2 Preliminaries

2.1 Bayesian Auctions

In a multi-parameter auction there are m items, denoted by $M = \{1, \ldots, m\}$, and n players, denoted by $N = \{1, \ldots, n\}$. Each player $i \in N$ has a non-negative value for each item $j \in M$, v_{ij}, which is independently drawn from distribution $\mathcal{D}_{ij}$. Player i's *true valuation* is $(v_{ij})_{j\in[m]}$. To simplify the notations, we may write v_i for $(v_{ij})_{j\in[m]}$ and v for $(v_i)_{i\in[n]}$. Letting $\mathcal{D}_i = \times_{j\in M}\mathcal{D}_{ij}$ and $\mathcal{D} = \times_{i\in N}\mathcal{D}_i$, we use $\mathcal{I} = (N, M, \mathcal{D})$ to denote the corresponding Bayesian auction instance and $OPT(\mathcal{I})$ the optimal BIC revenue of $\mathcal{I}$. When $\mathcal{I}$ is clear in the context, we write OPT for short.

In this work, we will consider several classes of multi-parameter auctions that are widely studied in the literature. A *single-item* auction has $m = 1$. A *unit-demand* auction is such that each player i's value for a subset S of items is $\max_{j\in S} v_{ij}$, so without loss of generality the seller allocates at most one item to each player. Finally, an *additive* auction is such that i's value for S is $\sum_{j\in S} v_{ij}$.

2.2 Query Complexity

In this work, we only allow the seller to access the prior distributions via two types of oracle queries: *value queries* and *quantile queries*. Given a distribution D over reals, in a value query, the seller sends a value $v \in \mathbb{R}$ and the oracle returns the corresponding quantile $q(v) \triangleq \Pr_{x\sim D}[x \geq v]$. In a quantile query, the seller sends a quantile $q \in [0, 1]$ and the oracle returns the corresponding value $v(q)$ such that $\Pr_{x\sim D}[x \geq v(q)] = q$. With *non-adaptive* queries, the seller first sends all his queries to the oracle, gets the answers back, and then runs the auction with the players. The *query complexity* is the number of queries made by the seller.

Note that the answer to a value query is unique. The quantile queries are a bit tricky, as for discrete distributions there may be multiple values corresponding to the same quantile q, or there may be none. When there are multiple values, to resolve the ambiguity, let the output of the oracle be the largest one: that is, $v(q) = \arg\max_z\{\Pr_{x\sim D}[x \geq z] = q\}$. When there is no value corresponding to q, the oracle returns the largest value whose corresponding quantile is larger than q: that is, $v(q) = \arg\max_z\{\Pr_{x\sim D}[x \geq z] > q\}$. So for any quantile query q, $v(q) = \arg\max_z\{\Pr_{x\sim D}[x \geq z] \geq q\}$ in general. Note that for any discrete distribution D and any quantile query $q > 0$, $v(q)$ is always in the support of D. When $q = 0$, $v(q)$ may be $+\infty$.

3 Lower Bounds

In this section, we prove lower bounds for the query complexity of Bayesian mechanisms, and we focus on DSIC mechanisms. Indeed, a BIC mechanism may not be BIC any more if the seller uses oracle queries to approximate the prior distribution $\mathcal{D}$, while a DSIC mechanism continues to be DSIC no matter which distribution the seller uses. As a building block for our general lower bound, we first prove the following for *single-item single-player* auctions.

Lemma 1. *For any constant $c > 1$, there exists a constant C such that, for any large enough H, any DSIC Bayesian mechanism $\mathcal{M}$ making less than $C\log_c H$ (randomized) non-adaptive value and quantile queries to the oracle, there exists a single-player single-item Bayesian auction instance $\mathcal{I} = (N, M, \mathcal{D})$ where the values are bounded in $[1, H]$, such that $Rev(\mathcal{M}(\mathcal{I})) < \frac{OPT(\mathcal{I})}{c}$.*

Proof. Given c, for any constant H, let $k \triangleq \lfloor \frac{1}{4}\log_{(8c)^{4c+2}} H\rfloor$. When H is large enough, we have

$$k = \lfloor \frac{\log_c H}{4(4c+2)\log_c(8c)} \rfloor \geq 1.$$

We divide the quantile interval $[0, 1]$ and the value interval $[1, H]$ into $k + 1$ sub-intervals, with their right-ends defined as follows: $q_k = 1$, $q_s = \frac{q_{s+1}}{(8c)^{4c+2}}$ for each $s \in \{k-1, \dots, 0\}$, $u_k = H$, and $u_s = \frac{u_{s+1}}{(4c)^{4c+2}}$ for each $s \in \{k-1, \dots, 0\}$. It is easy to see

$$q_0 = \frac{1}{(8c)^{k(4c+2)}} \geq H^{-\frac{1}{4}} \quad \text{and} \quad u_0 = \frac{H}{(4c)^{k(4c+2)}} \geq H \cdot q_0 \geq H^{\frac{3}{4}}.$$

From now on, we will ignore the intervals below u_0 and q_0.

Let $c' \triangleq 1 - \frac{1}{2c}$ and $C \triangleq \frac{1-c'}{8(4c+2)\log_c(8c)} = \frac{1}{16c(4c+2)\log_c(8c)}$. We have $C \log_c H < k(1-c')$. Accordingly, for any DSIC Bayesian mechanism $\mathcal{M}$ that makes less than $C \log_c H$ non-adaptive value and quantile queries, there exist a value interval (u_s, u_{s+1}) and a quantile interval (q_t, q_{t+1}) such that, with probability at least c', no value in (u_s, u_{s+1}) is queried and no quantile in (q_t, q_{t+1}) is queried either. Indeed, if this is not the case, then for any pair (u_s, u_{s+1}) and (q_t, q_{t+1}), with probability greater than $1 - c'$, either (u_s, u_{s+1}) is queried or (q_t, q_{t+1}) is queried. Since there are k value intervals and k quantile intervals, the expected total number of queries made by $\mathcal{M}$ is at least $k(1-c') > C \log_c H$, a contradiction.

We now construct $\lceil 4c \rceil$ different single-player single-item Bayesian instances $\{\mathcal{I}_z = (N, M, \mathcal{D}_z)\}_{z \in [\lceil 4c \rceil]}$, where the distributions outside the value range (u_s, u_{s+1}) and the quantile range (q_t, q_{t+1}) are all the same. Given such D_z's, with probability at least $c' = 1 - \frac{1}{2c}$ mechanism $\mathcal{M}$ cannot distinguish the $\mathcal{I}_z$'s from each other. We then show that when this happens, mechanism $\mathcal{M}$ cannot be a $2c$-approximation for all instances $\mathcal{I}_z$.

More precisely, the distribution $\mathcal{D}_z$ for each $z \in [\lceil 4c \rceil]$ is defined in Table 2 and illustrated in Figure 1 in Appendix A.1. Here δ is a small constant whose value will be determined in the analysis.

Value v_z	1	u_s	$(4c)^z u_s$	u_{s+1}
Probability of v_z	$1 - q_{t+1}$	δ	$q_{t+1} - q_t - 2\delta$	$q_t + \delta$

Table 2: Distribution $\mathcal{D}_z$.

It is easy to see that for each value query from $[1, u_s] \cup [u_{s+1}, H]$, the returned quantile is the same for all $\mathcal{D}_z$'s. Moreover, when a quantile query is from $[0, q_t] \cup [q_{t+1}, 1]$, the oracle's answer is again the same for all $\mathcal{D}_z$'s, as illustrated in Table 3 in Appendix A.2. Accordingly, with probability at least $1 - \frac{1}{2c}$, mechanism $\mathcal{M}$ cannot distinguish $\mathcal{D}_z$'s from each other, which means it cannot distinguish $\mathcal{I}_z$'s from each other.

We now analyze the optimal BIC revenue for those instances. For any $\mathcal{I}_z$, Myerson's mechanism is optimal: it sets a (randomized) threshold for the unique player, if the player bids at least the threshold then he gets the item and pays the threshold payment, otherwise the item is unsold. Letting $\delta \triangleq \frac{1}{H}$, it is not hard to verify that $OPT(\mathcal{I}_z) = (4c)^z u_s (q_{t+1} - \delta)$ for each $\mathcal{I}_z$.

Next, we analyze the revenue of $\mathcal{M}$. Since $\mathcal{M}$ is DSIC, the allocation rule must be monotone in the player's bid, and he will pay the threshold payment set by $\mathcal{M}$, denoted by P. Here P may also be randomized. Note that for all instances, setting $P < 4cu_s$ is strictly worse than setting $P = 4cu_s$, and setting $P > (4c)^{\lceil 4c \rceil} u_s$ is strictly worse than setting $P = (4c)^{\lceil 4c \rceil} u_s < u_{s+1}$. Also, for any instance $\mathcal{I}_z$ and any $z' \in \{1, \ldots, \lceil 4c \rceil - 1\}$, setting $P \in ((4c)^{z'} u_s, (4c)^{z'+1} u_s)$ is strictly worse than setting $P = (4c)^{z'+1} u_s$. Thus, when mechanism $\mathcal{M}$ cannot distinguish the $\mathcal{I}_z$'s, it must use the same P for all $\mathcal{I}_z$'s, and the best it can do is to set $P = (4c)^z u_s$ with some probability ρ_z for each $z \in [\lceil 4c \rceil]$. Because $\sum_{z \in [\lceil 4c \rceil]} \rho_z = 1$, there exists z^* such that $\rho_{z^*} \leq \frac{1}{4c}$. Thus we have

$$
\begin{aligned}
Rev(\mathcal{M}(\mathcal{I}_{z^*})) &\leq \frac{1}{4c} \cdot (4c)^{z^*} \cdot u_s \cdot (q_{t+1} - \delta) + (1 - \frac{1}{4c})(4c)^{z^*-1} \cdot u_s \cdot (q_{t+1} - \delta) \\
&< \frac{1}{2c} \cdot (4c)^{z^*} \cdot u_s \cdot (q_{t+1} - \delta) = \frac{1}{2c} OPT(\mathcal{I}_{z^*}),
\end{aligned}
$$

where the first inequality is because for any threshold other than $(4c)^{z^*} u_s$, the resulting expected revenue is no larger than that with the threshold being $(4c)^{z^*-1} u_s$. That is, when $\mathcal{M}$ cannot distinguish the $\mathcal{I}_z$'s, it cannot be a $2c$-approximation for $\mathcal{I}_{z^*}$.

As the revenue of $\mathcal{M}$ under $\mathcal{I}_{z^*}$ is at most $OPT(\mathcal{I}_{z^*})$ when it is able to distinguish $\mathcal{I}_{z^*}$ from all the other instances, we have

$$Rev(\mathcal{M}(\mathcal{I}_{z^*})) \leq (1 - \frac{1}{2c})\frac{1}{2c}OPT(\mathcal{I}_{z^*}) + \frac{1}{2c}OPT(\mathcal{I}_{z^*}) < \frac{1}{c}OPT(\mathcal{I}_{z^*}).$$

Thus $\mathcal{M}$ is not a c-approximation for $\mathcal{I}_{z^*}$, and Lemma 1 holds. □

Note that Lemma 1 applies to *every* constant approximation ratio $c > 1$. We extend this lemma to arbitrary multi-player multi-item Bayesian auctions with *succinct sub-additive* valuations, as follows, with the corresponding definitions and the proof of the theorem deferred to Appendix A.

Theorem 1. *For any constant $c > 1$, there exists a constant C such that, for any $n \geq 1, m \geq 1$, any large enough H, any succinct sub-additive valuation function profile $v = (v_i)_{i \in [n]}$, and any DSIC Bayesian mechanism $\mathcal{M}$ making less than $Cnm \log_c H$ non-adaptive value and quantile queries to the oracle, there exists a multi-item Bayesian auction instance $\mathcal{I} = (N, M, \mathcal{D})$ with valuation profile v, where $|N| = n, |M| = m$ and the item values are bounded in $[1, H]$, such that $Rev(\mathcal{M}(\mathcal{I})) < \frac{OPT(\mathcal{I})}{c}$.*

Succinct sub-additive auction is a very broad class and contains single-item, unit-demand, and additive auctions as special cases. Thus Theorem 1 automatically applies to those cases.

4 The Query Complexity for Bounded Distributions

In this section, we consider settings where all distributions are bounded within $[1, H]$, and we construct efficient query mechanisms whose query complexity matches our lower-bounds. We show that it is sufficient to use only value queries, and we define in Section 4.1 a universal query algorithm $\mathcal{A}_V$, which will be used as a black-box in our mechanisms. Given algorithm $\mathcal{A}_V$, the seller uses it to learn a distribution $\mathcal{D}' = \times_{i \in N, j \in M} \mathcal{D}'_{ij}$ that approximates the prior distribution $\mathcal{D}$ and is stochastically dominated by $\mathcal{D}$. The seller then runs existing DSIC Bayesian mechanisms using $\mathcal{D}'$. In this sense, all our mechanisms in this section are simple.

The multi-player single-item setting is already non-trivial, but still easy, since we have a good understanding of the optimal mechanism, which is Myerson's auction [26]. In particular, we make use of the *revenue monotonicity* theorem of [14]. However, the situation for unit-demand and additive auctions is much more subtle. The optimal auction could be very complicated and may involve lotteries and bundling, and revenue monotonicity does not necessarily hold [22]. Even (disregarding complexity issues and) assuming we can design an optimal mechanism for $\mathcal{D}'$, it is not clear at all how much revenue we can guarantee for the true distribution $\mathcal{D}$. To overcome this difficulty, we make use of recent developments on *simple* mechanisms with approximately optimal revenue.

The mechanism for unit-demand auctions is sequential post-price [24], thus the analysis is still relatively easy. For additive auctions, the simple mechanism is either running Myerson's auction separately for each item or running the VCG mechanism with a per-player entry fee [30, 5]. The Myerson's auction part is easy, but the VCG mechanism with entry fee is complicated. Via an easy and direct analysis, we lose a factor of m in the query complexity. In order to get the tight result, we need to really open the box of the analysis and do it differently in a number of places. Due to a lack of space, most proofs are provided in the appendix.

To sum up, our mechanisms are black-box reductions to known mechanisms, and thus simple, natural, and easy to implement in practice, while the analysis is non-black-box, non-trivial and interesting.

4.1 The Value-Query Algorithm

The query algorithm $\mathcal{A}_V$ is defined in Algorithm 1. Here $D \in \Delta(\mathbb{R})$ is the distribution to be queried. The algorithm takes two parameters, the value bound H and the precision factor $\delta > 0$, makes $O(\log_{1+\delta} H)$ value queries to the oracle, and then returns a discrete distribution D'. It is easy to verify that D' is stochastically dominated by D.

Algorithm 1 The Value-Query Algorithm $\mathcal{A}_V$

Input: The value bound H and the precision factor δ.

1: Let $k = \lceil \log_{1+\delta} H \rceil$ and define the *value vector* as $v = (v_0, v_1, \ldots, v_{k-1}, v_k) = (1, (1+\delta), (1+\delta)^2, \ldots, (1+\delta)^{k-1}, H)$.

2: Query the oracle for D with v, and receive a non-increasing quantile vector $q = (q(v_0), \cdots, q(v_k)) = (q_l)_{l \in \{0,\ldots,k\}}$. Note $q_0 = 1$.

3: Construct a discrete distribution D' as follows: $D'(v_l) = q_l - q_{l+1}$ for any $l \in \{0, \ldots, k\}$, where $q_{k+1} \triangleq 0$.

Output: Distribution D'.

4.2 Single-Item Auctions and Unit-Demand Auctions

Denoting by $\mathcal{M}_{MRS}$ Myerson's mechanism for single-item auctions, Mechanism 2 defines our *efficient value Myerson* mechanism $\mathcal{M}_{EVM}$.

Mechanism 2 Efficient Value Myerson Mechanism $\mathcal{M}_{EVM}$

1: Given the value bound H and a constant $\epsilon > 0$, run the value-query algorithm $\mathcal{A}_V$ with H and $\delta = \epsilon$ for each player i's distribution $\mathcal{D}_i$. Denote by $\mathcal{D}'_i$ the returned distribution. Let $\mathcal{D}' = \times_{i \in N} \mathcal{D}'_i$.

2: Run $\mathcal{M}_{MRS}$ with $\mathcal{D}'$ and the players' reported values, $b = (b_i)_{i \in N}$, to get allocation $x = (x_i)_{i \in N}$ and price profile $p = (p_i)_{i \in N}$ as the outcome.

It is easy to see that the query complexity of $\mathcal{M}_{EVM}$ is $O(n \log_{1+\epsilon} H)$, since each distribution $\mathcal{D}_i$ needs $O(\log_{1+\epsilon} H)$ value queries in $\mathcal{A}_V$. Note that when ϵ is sufficiently small, $O(n \log_{1+\epsilon} H) \approx O(n\epsilon^{-1} \log H)$. It is also immediate that $\mathcal{M}_{EVM}$ is DSIC.

In this section and throughout the paper, we often need to analyze "mismatching" cases where a Bayesian mechanism $\mathcal{M}$ uses distribution $\mathcal{D}'$ while the actual Bayesian instance is $\mathcal{I} = (N, M, \mathcal{D})$ (i.e., the players' true values are drawn from $\mathcal{D}$). We use $Rev(\mathcal{M}(\mathcal{I}; \mathcal{D}'))$ to denote the expected revenue in this case. By construction, $Rev(\mathcal{M}_{EVM}(\mathcal{I})) = Rev(\mathcal{M}_{MRS}(\mathcal{I}; \mathcal{D}'))$.

Because the $\mathcal{D}'$ constructed in $\mathcal{M}_{EVM}$ is stochastically dominated by $\mathcal{D}$, letting $\mathcal{I}' = (N, M, \mathcal{D}')$ be the instance under $\mathcal{D}'$, by revenue monotonicity we have $Rev(\mathcal{M}_{MRS}(\mathcal{I}; \mathcal{D}')) \geq Rev(\mathcal{M}_{MRS}(\mathcal{I}'))$. By Lemma 5 of [14], $Rev(\mathcal{M}_{MRS}(\mathcal{I}')) \geq \frac{OPT(\mathcal{I})}{1+\epsilon}$. Thus we have proved the following.

Theorem 2. $\forall \epsilon > 0$*, for any single-item instance* $\mathcal{I} = (N, M, \mathcal{D})$ *with values bounded within* $[1, H]$*, mechanism* $\mathcal{M}_{EVM}$ *is DSIC, has query complexity* $O(n \log_{1+\epsilon} H)$*, and* $Rev(\mathcal{M}_{EVM}(\mathcal{I})) \geq \frac{OPT(\mathcal{I})}{1+\epsilon}$.

The construction for unit-demand auctions is similar, except the seller uses as a blackbox the DSIC mechanism of [24], denoted by $\mathcal{M}_{UD}$. See Mechanism 3 for the resulting mechanism $\mathcal{M}_{EVUD}$.

The main difficulty for unit-demand auctions is that we do not have the revenue monotonicity theorem as in single-item auctions. Accordingly, our analysis comes in a non-blackbox way and relies on the *COPIES setting* [8, 24], which provides an upper-bound for the optimal BIC revenue.

Mechanism 3 Mechanism $\mathcal{M}_{EVUD}$ for Unit-Demand Auctions

1: Given H and $\epsilon > 0$, run the value-query algorithm $\mathcal{A}_V$ with H and $\delta = \epsilon$ for each player i's distribution $\mathcal{D}_{ij}$ for each item j. Denote by $\mathcal{D}'_{ij}$ the returned distribution. Let $\mathcal{D}'_i = \times_{j\in M}\mathcal{D}'_{ij}$ and $\mathcal{D}' = \times_{i\in N}\mathcal{D}'_i$.

2: Run $\mathcal{M}_{UD}$ with $\mathcal{D}'$ and the players' reported values, $b = (b_{ij})_{i\in N, j\in M}$, to get allocation $x = (x_{ij})_{i\in N, j\in M}$ and price profile $p = (p_i)_{i\in N}$ as the outcome.

By properly upper-bounding the revenue in the COPIES setting under $\mathcal{D}'$, we are able to upper-bound the optimal BIC revenue using the expected revenue of $\mathcal{M}_{EVUD}$. More precisely, we have the following theorem, proved in Appendix B.1.

Theorem 3. $\forall \epsilon > 0$*, for any unit-demand instance* $\mathcal{I} = (N, M, \mathcal{D})$ *with values in* $[1, H]$*, mechanism* $\mathcal{M}_{EVUD}$ *is DSIC, has query complexity* $O(mn \log_{1+\epsilon} H)$*, and* $Rev(\mathcal{M}_{EVUD}(\mathcal{I})) \geq \frac{OPT(\mathcal{I})}{24(1+\epsilon)}$.

Letting $c = 24(1 + \epsilon)$, we have the query complexity in Table 1.

4.3 Additive Auctions

For additive auctions, denote by $\mathcal{M}_A$ the DSIC Bayesian mechanism in [30, 5]. For any Bayesian instance $\mathcal{I} = (N, M, \mathcal{D})$, this mechanism is such that the seller chooses between two mechanisms, whichever generates higher expected revenue. The first is the "individual Myerson" mechanism, denoted by $\mathcal{M}_{IM}$, which sells each item separately using Myerson's mechanism. The second is the VCG mechanism with optimal per-player entry fees, denoted by $\mathcal{M}_{BVCG}$.

In our mechanism $\mathcal{M}_{EVA}$ that approximates mechanism $\mathcal{M}_A$ using value queries, the seller queries about $\mathcal{D}$ using algorithm $\mathcal{A}_V$ with parameters different from before. Given the resulting distribution $\mathcal{D}'$, the seller either runs $\mathcal{M}_{IM}$ or runs $\mathcal{M}_{BVCG}$ as a blackbox, resulting in query mechanisms $\mathcal{M}_{EVIM}$ and $\mathcal{M}_{EVBVCG}$. We only define the latter in Mechanism 4, and the former simply replaces $\mathcal{M}_{BVCG}$ with $\mathcal{M}_{IM}$. Note that $Rev(\mathcal{M}_{EVIM}(\mathcal{I})) = Rev(\mathcal{M}_{IM}(\mathcal{I}; \mathcal{D}'))$ and $Rev(\mathcal{M}_{EVBVCG}(\mathcal{I})) = Rev(\mathcal{M}_{BVCG}(\mathcal{I}; \mathcal{D}'))$. However, the seller cannot compute these two revenue and choose the better one, because he does not know $\mathcal{D}$. Thus, we let the seller randomly choose between the two mechanisms, according to proper probabilities that we will define in the analysis. We have the following theorem, proved in Appendix B.2.

Mechanism 4 Mechanism $\mathcal{M}_{EVBVCG}$ to Approximate $\mathcal{M}_{BVCG}$ via Value Queries

1: Given H and $\epsilon > 0$, run the value-query algorithm $\mathcal{A}_V$ with H and $\delta = \sqrt{\epsilon + 1} - 1$ for each player i's distribution $\mathcal{D}_{ij}$ for each item j. Denote by $\mathcal{D}'_{ij}$ the returned distribution. Let $\mathcal{D}'_i = \times_{j\in M}\mathcal{D}'_{ij}$ and $\mathcal{D}' = \times_{i\in N}\mathcal{D}'_i$.

2: Run $\mathcal{M}_{BVCG}$ with $\mathcal{D}'$ and the players' reported values, $b = (b_{ij})_{i\in N, j\in M}$, to get allocation $x = (x_{ij})_{i\in N, j\in M}$ and price profile $p = (p_i)_{i\in N}$ as the outcome.

Theorem 4. $\forall \epsilon > 0$*, for any additive instance* $\mathcal{I} = (N, M, \mathcal{D})$ *with values in* $[1, H]$*, mechanism* $\mathcal{M}_{EVA}$ *is DSIC, has query complexity* $O(mn \log_{1+\epsilon} H)$*, and* $Rev(\mathcal{M}_{EVA}(\mathcal{I})) \geq \frac{OPT(\mathcal{I})}{8(1+\epsilon)}$.

Letting $c = 8(1 + \epsilon)$, we have the query complexity in Table 1. The proof of Theorem 4 is much harder than single-item and unit-demand auctions. Not only revenue monotonicity may not hold, but also the revenue of additive auctions may not be bounded by the COPIES setting. Instead, our analysis is based on the duality framework in [5], properly revised for our setting. Below we briefly discuss the main ideas.

Proof Ideas: As in [5], we only need to consider the prior distribution $\mathcal{D}$ with finite support. Let V_{ij} be the support of $\mathcal{D}_{ij}$ for each player i and item j, $V_i = \times_{j\in M} V_{ij}$ and $V = \times_{i\in N} V_i$. In the optimal BIC mechanism, when player i bids v_i, let $\pi_{ij}(v_i)$ be the probability for him to get item j and $p_i(v_i)$ be his expected payment, taken over the randomness of the other players' values and the randomness of the mechanism. Let $\pi = (\pi_{ij}(v_i))_{i\in N, j\in M, v_i\in V_i}$ and $p = (p_i(v_i))_{i\in N, v_i \in V_i}$. The pair (π, p) is called the *reduced form* (of the optimal BIC mechanism) [3].

Denote by $\tilde{\varphi}_{ij}(v_{ij})$ Myerson's (ironed) virtual value when player i's value on item j is v_{ij}. For any value sub-profile v_{-i} of the players other than i, let $\beta_{ij}(v_{-i}) = \max_{i'\neq i} v_{i'j}$: that is, the highest bid on item j excluding player i. Moreover, let $r_{ij}(v_{-i}) = \max_{x\geq \beta_{ij}(v_{-i})}\{x\cdot \Pr_{v_{ij}\sim\mathcal{D}_{ij}}[v_{ij}\geq x]\}$, $r_i(v_{-i}) = \sum_j r_{ij}(v_{-i})$, $r_i = \mathbb{E}_{v_{-i}\sim\mathcal{D}_{-i}}[r_i(v_{-i})]$, and finally $r=\sum_i r_i$. Note that r is the expected revenue by running the 1-look-ahead mechanism of [27] for each item separately, and $r \leq Rev(\mathcal{M}_{IM}(\mathcal{I}))$.

Next, we use a different method from [5] to partition each player i's value space V_i into $m+1$ subsets. More precisely, given $\delta > 0$ and v_{-i}, let $R_0^{(v_{-i})} = \{v_i \in V_i \mid v_{ij} < (1+\delta)\beta_{ij}(v_{-i}), \forall j\}$. For any $v_i \notin R_0^{(v_{-i})}$, let $j = \arg\max\{v_{ij} - (1+\delta)\beta_{ij}(v_{-i})\}$ with ties broken lexicographically, and add v_i to the set $R_j^{(v_{-i})}$: note that $v_{ij} - (1+\delta)\beta_{ij}(v_{-i}) \geq 0$ in this case. Similar to Theorem 3 of [5], the optimal BIC revenue can be upper-bounded by the sum of the following terms, where $\mathcal{D}_i(v_i)$ and $\mathcal{D}_{-i}(v_{-i})$ are respectively the probabilities of v_i and v_{-i} under $\mathcal{D}$, and $\mathbf{I}$ is the indicator function.

$$
\begin{aligned}
OPT(\mathcal{I}) \;\leq\; & \sum_{i\in N}\sum_{v_i\in V_i}\sum_{j\in M}\mathcal{D}_i(v_i)\pi_{ij}(v_i)\left(v_{ij}\cdot \Pr_{v_{-i}\sim\mathcal{D}_{-i}}[v_i\notin R_j^{(v_{-i})}] + \tilde{\varphi}_{ij}(v_{ij})\cdot \Pr_{v_{-i}\sim\mathcal{D}_{-i}}[v_i \in R_j^{(v_{-i})}]\right) \\
\leq\; & \sum_{i}\sum_{v_i\in V_i}\sum_{j}\mathcal{D}_i(v_i)\cdot\pi_{ij}(v_i)\cdot\tilde{\varphi}_{ij}(v_{ij})\cdot\Pr_{v_{-i}\sim\mathcal{D}_{-i}}[v_i\in R_j^{(v_{-i})}] \qquad \text{(Single)} \\
& + \sum_{i}\sum_{v_i\in V_i}\sum_{j}\mathcal{D}_i(v_i)\cdot\pi_{ij}(v_i)\cdot\sum_{v_{-i}\in V_{-i}} v_{ij}\cdot\mathcal{D}_{-i}(v_{-i})\mathbf{I}_{v_{ij}<(1+\delta)\beta_{ij}(v_{-i})} \qquad \text{(Under)} \\
& + \sum_{i}\sum_{v_i\in V_i}\sum_{j}\mathcal{D}_i(v_i)\cdot\pi_{ij}(v_i)\cdot\sum_{v_{-i}\in V_{-i}} (1+\delta)\beta_{ij}(v_{-i})\mathcal{D}_{-i}(v_{-i})\mathbf{I}_{v_{ij}\geq(1+\delta)\beta_{ij}(v_{-i})} \qquad \text{(Over)} \\
& + \sum_{i}\sum_{v_{-i}\in V_{-i}}\mathcal{D}_{-i}(v_{-i})\sum_{j}\sum_{v_{ij}>(1+\delta)\beta_{ij}(v_{-i})+r_i(v_{-i})}\mathcal{D}_{ij}(v_{ij})\cdot(v_{ij}-(1+\delta)\beta_{ij}(v_{-i})) \\
& \qquad \cdot\Pr_{v_{i,-j}\sim\mathcal{D}_{i,-j}}[\exists k\neq j, v_{ik}-(1+\delta)\beta_{ik}(v_{-i})\geq v_{ij}-(1+\delta)\beta_{ij}(v_{-i})] \qquad \text{(Tail)} \\
& + \sum_{i}\sum_{v_{-i}\in V_{-i}}\mathcal{D}_{-i}(v_{-i})\sum_{j}\sum_{(1+\delta)\beta_{ij}(v_{-i})\leq v_{ij}\leq(1+\delta)\beta_{ij}(v_{-i})+r_i(v_{-i})}\mathcal{D}_{ij}(v_{ij}) \\
& \qquad \cdot(v_{ij}-(1+\delta)\beta_{ij}(v_{-i})). \qquad \text{(Core)} \qquad (1)
\end{aligned}
$$

For the terms Single, Under, Over and Tail, we are able to upper-bound them using $Rev(\mathcal{M}_{EVIM}(\mathcal{I})) = Rev(\mathcal{M}_{IM}(\mathcal{I};\mathcal{D}'))$, losing only an extra $O(\delta)$ fraction in revenue. The Core part is the most complicated, and we use $\mathcal{M}_{EVBVCG}$ and $\mathcal{M}_{EVIM}$ together to upper-bound it. Below we only introduce the main ideas for bounding the Core. All the details are explained in Appendix B.

First, we show that

$$Rev(\mathcal{M}_{EVBVCG}(\mathcal{I})) = Rev(\mathcal{M}_{BVCG}(\mathcal{I};\mathcal{D}')) \geq \sum_i \mathbb{E}_{v_{-i}\sim\mathcal{D}_{-i}}\mathbb{E}_{v_i'\sim\mathcal{D}_i'} Rev(\mathcal{M}_{BVCG}(v_i', v_{-i};\mathcal{D}')). \quad (2)$$

Second, we provide a new entry fee $e_i'(v_{-i})$ to lower-bound Equation 2, such that for any player i with $v_i'\sim\mathcal{D}_i'$, and any v_{-i}, player i accepts $e_i'(v_{-i})$ with probability at least $\frac{1}{2}$. Therefore,

$$\sum_i \mathbb{E}_{v_{-i}\sim\mathcal{D}_{-i}}\mathbb{E}_{v_i'\sim\mathcal{D}_i'} Rev(\mathcal{M}_{BVCG}(v_i', v_{-i};\mathcal{D}')) \geq \frac{1}{2}\sum_i\sum_{v_{-i}\in V_{-i}}\mathcal{D}_{-i}(v_{-i})\cdot e_i'(v_{-i}).$$

Finally, combining the precise formula of $e'_i(v_{-i})$ with the Core, we show that

$$\text{Core} \leq 2(1+\delta)[Rev(\mathcal{M}_{EVBVCG}(\mathcal{I})) + Rev(\mathcal{M}_{EVIM}(\mathcal{I}))].$$

Combing all our upper-bounds for Single, Under, Over, Tail and Core gives the desired revenue bound in Theorem 4. □

5 The Query Complexity for Unbounded Distributions

Next, we construct efficient query mechanisms for arbitrary distributions whose supports can be unbounded. For a mechanism to approximate the optimal Bayesian revenue using *finite* queries to such distributions, it is intuitive that some kind of *small-tail assumption* for the distributions is needed. Indeed, given any mechanism with query complexity C, there always exists a distribution that has a sufficiently small probability mass around a sufficiently large value, such that the mechanism cannot find it using C queries. If this probability mass is where all the revenue comes (e.g., all the remaining probability mass is around value 0), then the mechanism cannot be a good approximation. Accordingly, following the literature, we assume the expected revenue generated from the "tail" of the distributions is negligible compared to the optimal revenue; see Section 5.1.

For unbounded distributions, even with small-tail assumptions, it is hard to generate good revenue with finite *value* queries. In fact, we show it is sufficient to use only *quantile* queries. As before, the seller uses our quantile-query algorithm $\mathcal{A}_Q$ (defined in Section 5.2) to learn a distribution $\mathcal{D}'$ that approximates $\mathcal{D}$, and then reduces to simple mechanisms under $\mathcal{D}'$. However, even for single-item auctions, the analysis under quantile queries is not so simple. Indeed, under value queries, it is easy to "under-price" the item so that the probability of sale is the same as in the optimal mechanism for $\mathcal{D}$. Under quantile queries, unfortunately under-pricing may lose a large amount of revenue, because there is no guarantee on where the values are for given quantiles. Instead, the main idea in using quantile queries is to *"over-price"* the item. This is risky in many auction design scenarios, because it may significantly reduce the probability of sale, and thus lose a lot of revenue. We prove a key technical lemma in Lemma 2, where we show that by discretizing the quantile space properly, we can over-price the item while almost preserving the probability of sale as in the optimal mechanism under $\mathcal{D}$. In Lemma 4 of Appendix C, we prove another technical lemma showing that proper over-pricing can also be done in additive auctions.

5.1 Small-Tail Assumptions

A Bayesian auction instance $\mathcal{I}$ satisfies the *Small-Tail Assumption 1* if there exists a function[1] $h : (0,1) \to (0,1)$ such that, for any constant $\delta_1 \in (0,1)$ and any BIC mechanism $\mathcal{M}$, letting $\epsilon_1 = h(\delta_1)$, we have

$$\mathop{\mathbb{E}}_{v\sim\mathcal{D}} \mathbf{I}_{\exists i,j,q_{ij}(v_{ij})\leq\epsilon_1} Rev(\mathcal{M}(v;\mathcal{I})) \leq \delta_1 OPT(\mathcal{I}). \tag{3}$$

Here $q_{ij}(v_{ij})$ is the quantile of v_{ij} under distribution $\mathcal{D}_{ij}$, $Rev(\mathcal{M}(v;\mathcal{I}))$ is the revenue of $\mathcal{M}$ under the Bayesian instance $\mathcal{I}$ when the true valuation profile is v, and $\mathbf{I}$ is the indicator function. For discrete distributions, Equation 3 is imposed on the ϵ_1 probability mass over the highest values.

Equation 3 immediately implies the following weaker *Small-Tail Assumption 2*: there exists a function $h : (0,1) \to (0,1)$ such that, for any constant $\delta_1 \in (0,1)$, letting $\epsilon_1 = h(\delta_1)$, we have

$$\mathop{\mathbb{E}}_{v\sim\mathcal{D}} \mathbf{I}_{\exists i,j,q_{ij}(v_{ij})\leq\epsilon_1} Rev_{OPT}(v;\mathcal{I}) < \delta_1 OPT(\mathcal{I}). \tag{4}$$

[1] If computation complexity is a concern, then one can further require that the function is efficiently computable.

Here $Rev_{OPT}(v; \mathcal{I})$ is the revenue generated by the optimal BIC mechanism for $\mathcal{I}$ when the true valuation profile is v. Note that both small-tail assumptions are naturally satisfied when the distributions have bounded supports.

5.2 The Quantile-Query Algorithm

We define our quantile-query algorithm $\mathcal{A}_Q$ in Algorithm 5. As before, $D \in \Delta(\mathbb{R})$ is the distribution to be queried. The algorithm takes two parameters, the tail length ϵ_1 and the precision factor δ, makes $O(\log_{1+\delta} \frac{1}{\epsilon_1})$ quantile queries to the oracle, and then returns a discrete distribution D'.

Algorithm 5 The Quantile-Query Algorithm $\mathcal{A}_Q$

Input: the tail length ϵ_1 and the precision factor δ.

1: Let $k = \lceil \log_{1+\delta} \frac{1}{\epsilon_1} \rceil$ and define the *quantile vector* as $q = (q_0, q_1, \ldots, q_{k-1}, q_k) = (1, \epsilon_1(1+\delta)^{k-1}, \ldots, \epsilon_1(1+\delta), \epsilon_1)$.

2: Query the oracle for D with q, and receive a non-decreasing value vector $(v_l)_{l \in \{0,\ldots,k\}}$.

3: Construct a distribution D' as follows: $D'(v_l) = q_l - q_{l+1}$ for each $l \in \{0, \ldots, k\}$, where $q_{k+1} = 0$.

Output: Distribution D'.

5.3 Single-Item Auctions

Mechanism 6 defines our *efficient quantile Myerson* mechanism $\mathcal{M}_{EQM}$, and we have the following.

Mechanism 6 Efficient Quantile Myerson Mechanism $\mathcal{M}_{EQM}$

1: Given $\epsilon > 0$, run algorithm $\mathcal{A}_Q$ with $\delta = \frac{\epsilon}{3}$ and $\epsilon_1 = h(\frac{2\epsilon}{3(1+\epsilon)})$ (i.e., $\delta_1 = \frac{2\epsilon}{3(1+\epsilon)}$ for Small Tail Assumption 2), for each player i's distribution $\mathcal{D}_i$. Denote by $\mathcal{D}'_i$ the returned distribution. Let $\mathcal{D}' = \times_{i \in N} \mathcal{D}'_i$.

2: Run Myerson's mechanism $\mathcal{M}_{MRS}$ with $\mathcal{D}'$ and the players' reported values, $b = (b_i)_{i \in N}$, to get allocation $x = (x_i)_{i \in N}$ and price profile $p = (p_i)_{i \in N}$ as the outcome.

Theorem 5. *$\forall \epsilon > 0$, any single-item instance $\mathcal{I} = (N, M, \mathcal{D})$ satisfying Small-Tail Assumption 2, $\mathcal{M}_{EQM}$ is DSIC, has query complexity $O(-n \log_{1+\frac{\epsilon}{3}} h(\frac{2\epsilon}{3(1+\epsilon)}))$, and $Rev(\mathcal{M}_{EQM}(\mathcal{I})) \geq \frac{OPT(\mathcal{I})}{1+\epsilon}$.*

Recall $\mathcal{I}' = (N, M, \mathcal{D}')$ is the instance under $\mathcal{D}'$. We first prove the following key lemma, which is analogous to Lemma 5 of [14], but is for unbounded distributions and the quantile space.

Lemma 2. *$Rev(\mathcal{M}_{MRS}(\mathcal{I}')) \geq \frac{1}{1+\epsilon} OPT(\mathcal{I})$.*

Proof. For each player i, denote the support of $\mathcal{D}'_i$ by $V'_i = (v'_{i;l})_{l \in \{0,\ldots,k\}}$. We first define a way to couple the values $v'_i \sim \mathcal{D}'_i$ with the values $v_i \sim \mathcal{D}_i$.

On the one hand, for any value $v_i \geq v'_{i;0}$, let v_i^- be v_i *rounded down* to the support of $\mathcal{D}'_i$, such that v_i^- is distributed according to $\mathcal{D}'_i$ whenever v_i is distributed according to $\mathcal{D}_i$. Recall that under value queries, v_i^- is simply the largest value in V'_i that is less than or equal to v_i, no matter whether $\mathcal{D}_i$ is continuous or discrete. Under quantile queries, when $\mathcal{D}_i$ is continuous, the same deterministic round-down scheme still works. However, the situation is more subtle when $\mathcal{D}_i$ is discrete, and we need a randomized round-down scheme to ensure the relationship between v_i and v_i^-. More precisely, by the definition of quantile queries, V'_i is a subset of $\mathcal{D}_i$'s support. If v_i is not in V'_i, then it is still deterministically rounded down as before. If v_i is in V'_i, say $v_i = v'_{i;l}$, then by the definition of quantile queries and the construction of $\mathcal{D}'_i$, we have $\Pr_{x\sim\mathcal{D}_i}[x \geq v_i] \geq q_l = \Pr_{x\sim\mathcal{D}'_i}[x \geq v_i]$. In this case, v_i is rounded down to $v'_{i;l-1}$ (i.e., $v_i^- = v'_{i;l-1}$) with probability

$$\frac{\Pr_{x\sim\mathcal{D}_i}[x \geq v_i] - \Pr_{x\sim\mathcal{D}'_i}[x \geq v_i]}{\mathcal{D}_i(v_i)},$$

and to $v'_{i;l}$ (i.e., $v_i^- = v'_{i;l}$) with probability

$$1 - \frac{\Pr_{x\sim\mathcal{D}_i}[x \geq v_i] - \Pr_{x\sim\mathcal{D}'_i}[x \geq v_i]}{\mathcal{D}_i(v_i)}.$$

Following this scheme, it is not hard to verify that $\Pr_{v_i\sim\mathcal{D}_i}[v_i^- \geq v'_{i;l}] = q_l$ for any $l \in \{0, \ldots, k\}$, thus v_i^- is distributed according to $\mathcal{D}'_i$, as desired.

No matter what v_i^- is, let v_i^+ be the smallest value in V'_i that is strictly larger than v_i^- (if no such value exists, then $v_i^+ = +\infty$). That is, $v_i^+ \geq v_i$ and v_i^+ is v_i "rounded up", which was not needed under value queries and is new for quantile queries.

On the other hand, for any value $v'_i \sim \mathcal{D}'_i$, let v_i be *resampled* from $\mathcal{D}_i$ conditional on "v_i rounded down to v'_i", so that v_i is distributed according to $\mathcal{D}_i$ whenever v'_i is distributed according to $\mathcal{D}'_i$. Again, under value queries, the resampling is simply conditional on $v_i \in [v'_{i;l}, v'_{i;l+1})$ when $v'_i = v'_{i;l}$, no matter whether $\mathcal{D}_i$ is continuous or discrete. Under quantile queries, this resampling scheme still works when $\mathcal{D}_i$ is continuous. When $\mathcal{D}_i$ is discrete, we need to "undo" the randomized round-down scheme defined above. More precisely, letting $v'_i = v'_{i;l}$, v_i is set ot be $v'_{i;l+1}$ with probability

$$p_1 = \frac{\Pr_{x\sim\mathcal{D}_i}[x \geq v'_{i;l+1}] - q_{l+1}}{\mathcal{D}'_i(v'_{i;l})};$$

is resampled from $\mathcal{D}_i$ conditional on $v_i \in (v'_{i;l}, v'_{i;l+1})$ with probability

$$p_2 = \frac{\Pr_{x\sim\mathcal{D}_i}[v'_{i;l} < x < v'_{i;l+1}]}{\mathcal{D}'_i(v'_{i;l})};$$

and is set to be $v'_{i;l}$ with probability

$$p_3 = \frac{\mathcal{D}_i(v'_{i;l}) - \Pr_{x\sim\mathcal{D}_i}[x \geq v'_{i;l}] + q_l}{\mathcal{D}'_i(v'_{i;l})}.$$

Following this resampling scheme, it is not hard to verify that v_i is distributed according to $\mathcal{D}_i$ whenever v'_i is distributed according to $\mathcal{D}'_i$.

Given the round-down and the resampling schemes above, we consider the Bayesian mechanism $\mathcal{M}^*$ defined in Mechanism 7 for $\mathcal{I}'$, and compare its revenue with that of $\mathcal{M}_{MRS}$. We first claim that $\mathcal{M}^*$ is a DSIC mechanism, which is proved in Appendix C.1.

Claim 1. *$\mathcal{M}^*$ is DSIC.*

To analyze the revenue of $\mathcal{M}^*$, note that by construction, when v'_i is distributed according to $\mathcal{D}'_i$, the resampled v_i in $\mathcal{M}^*$ is distributed according to $\mathcal{D}_i$. Moreover, each v'_i is distributed as if we first sample v_i from $\mathcal{D}_i$ and then setting $v'_i = v_i^-$.

Thus, mechanism $\mathcal{M}^*$ on instance $\mathcal{I}'$ essentially generates the same expected revenue as $\mathcal{M}_{MRS}$ on instance $\mathcal{I}$, except for the case when $v'_i < p_i \leq v_i$ for the winner i. Fortunately, we are able to upper-bound the probability of this event and thus upper-bound the expected revenue loss. More

Mechanism 7 A Bayesian mechanism $\mathcal{M}^*$ for instance $\mathcal{I}'$

1: Each player i reports his value v'_i, and the mechanism discards the report that is not in V'_i.
2: For each player i, generate value v_i according to v'_i using our resampling scheme.
3: Run $\mathcal{M}_{MRS}$ with the value profile v and the prior distribution $\mathcal{D}$, to get the price p_i and the allocation $x_i \in \{0,1\}$ for each player i.
4: If $x_i = 1$ and $p_i \leq v'_i$, sell the item to i and charge him p_i; otherwise, set $x_i = 0$ and $p_i = 0$.

precisely, for each player i, we write p_i as $p_i(v_{-i}; \mathcal{D})$ to emphasize that it is the threshold payment for i given v_{-i} and $\mathcal{D}$, and does not depend on v_i or v'_i. We have

$$\begin{aligned} Rev(\mathcal{M}^*(\mathcal{I}')) &= \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} p_i(v_{-i};\mathcal{D}) \mathbf{I}_{v_i^- \geq p_i(v_{-i};\mathcal{D})} \\ &= \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} p_i(v_{-i};\mathcal{D}) \cdot \Pr_{v_i\sim\mathcal{D}_i}[v_i^- \geq p_i(v_{-i};\mathcal{D})]. \end{aligned} \tag{5}$$

Here the first equality holds because of the relationship between $\mathcal{D}'$ and $\mathcal{D}$ as established by our rounding and resampling schemes, and because each player i in $\mathcal{M}^*$ pays the same threshold price as in mechanism $\mathcal{M}_{MRS}$ whenever v'_i is at least the threshold, and pays 0 otherwise. By the construction of the distribution $\mathcal{D}'$, we have the following claim, proved in Appendix C.1.

Claim 2. $\Pr_{v_i\sim\mathcal{D}_i}[v_i \geq p_i(v_{-i};\mathcal{D}) | q_i(v_i) > \epsilon_1] \leq (1+\delta) \Pr_{v_i\sim\mathcal{D}_i}[v_i^- \geq p_i(v_{-i};\mathcal{D})]$.

Combining Equation 5, Claim 2 and Small Tail Assumption 2, we are able to lower-bound the revenue of $\mathcal{M}^*$ as follows, which is also proved in Appendix C.1.

Claim 3. *$Rev(\mathcal{M}^*(\mathcal{I}')) \geq \frac{1}{1+\epsilon} OPT(\mathcal{I})$.*

By the optimality of $\mathcal{M}_{MRS}$, $Rev(\mathcal{M}_{MRS}(\mathcal{I}')) \geq Rev(\mathcal{M}^*(\mathcal{I}'))$, and Lemma 2 holds. □

Proof of Theorem 5. First, mechanism $\mathcal{M}_{EQM}$ is DSIC because $\mathcal{M}_{MRS}$ is DSIC. Second, it is easy to see that the query complexity of $\mathcal{M}_{EQM}$ is $O(-n \log_{1+\frac{\epsilon}{3}} h(\frac{2\epsilon}{3(1+\epsilon)}))$, because there are $k+1 = \lceil \log_{1+\frac{\epsilon}{3}} \frac{1}{h(\frac{2\epsilon}{3(1+\epsilon)})} \rceil + 1$ quantile queries for each player and there are n players in total. By definition, $Rev(\mathcal{M}_{EQM}(\mathcal{I})) = Rev(\mathcal{M}_{MRS}(\mathcal{I};\mathcal{D}'))$. By construction, $\mathcal{D}'$ is stochastically dominated by $\mathcal{D}$. Thus by the revenue monotonicity theorem of [14], $Rev(\mathcal{M}_{MRS}(\mathcal{I};\mathcal{D}')) \geq Rev(\mathcal{M}_{MRS}(\mathcal{I}'))$. Combining these two equations with Lemma 2, Theorem 5 holds. □

Mechanism $\mathcal{M}_{EQM}$ and Theorem 5 immediately extend to single-parameter downward-closed settings. Finally, when the distributions are regular, we are able to prove an even better query complexity and a matching lower-bound; see Section 7.

5.4 Unit-Demand Auctions

The unit-demand mechanism $\mathcal{M}_{EQUD}$ is very similar (see Mechanism 8), and we have the following.

Theorem 6. *$\forall \epsilon > 0$, any unit-demand instance $\mathcal{I} = (N, M, \mathcal{D})$ satisfying Small-Tail Assumption 2, $\mathcal{M}_{EQUD}$ is DSIC, has query complexity $O(-mn \log_{1+\frac{\epsilon}{3}} h(\frac{2\epsilon}{3(1+\epsilon)}))$, and $Rev(\mathcal{M}_{EQUD}(\mathcal{I})) \geq \frac{OPT(\mathcal{I})}{24(1+\epsilon)}$.*

The proof of Theorem 6 is similar to that of Theorem 3, except that we replace the use of Lemma 5 of [14] by our Lemma 2, and the round-down scheme is replaced by the randomized round-down scheme designed in the proof of Lemma 2. Thus the details have been omitted.

Mechanism 8 Mechanism $\mathcal{M}_{EQUD}$ for Unit-Demand Auctions

1: Given $\epsilon > 0$, run algorithm $\mathcal{A}_Q$ with $\delta = \frac{\epsilon}{3}$ and $\epsilon_1 = h(\frac{2\epsilon}{3(1+\epsilon)})$ (i.e., $\delta_1 = \frac{2\epsilon}{3(1+\epsilon)}$ for Small Tail Assumption 2), for each player i's distribution $\mathcal{D}_{ij}$ on each item j. Denote by $\mathcal{D}'_{ij}$ the returned distribution. Let $\mathcal{D}'_i = \times_{j\in M}\mathcal{D}'_{ij}$ and $\mathcal{D}' = \times_{i\in N}\mathcal{D}'_i$.

2: Run mechanism $\mathcal{M}_{UD}$ with $\mathcal{D}'$ and the players' reported values, $b = (b_{ij})_{i\in N, j\in M}$, to get allocation $x = (x_{ij})_{i\in N, j\in M}$ and price profile $p = (p_i)_{i\in N}$ as the outcome.

5.5 Additive Auctions

For additive auctions, we cannot use Small-Tail Assumption 2, because it does not imply that the revenue loss on the tail by running $\mathcal{M}_{BVCG}$ is much less than the revenue of the optimal mechanism. To approximate $\mathcal{M}_{BVCG}$, not only we need Small-Tail Assumption 1, but we also approximate $\mathcal{D}$ by running the quantile-query algorithm $\mathcal{A}_Q$ with different parameters. The resulting mechanism $\mathcal{M}_{EQBVCG}$ is defined in Mechanism 9, and the mechanism $\mathcal{M}_{EQIM}$ simply replaces $\mathcal{M}_{BVCG}$ with $\mathcal{M}_{IM}$. Again, since the seller cannot compute the expected revenue of query mechanisms without knowing $\mathcal{D}$, in the final mechanism $\mathcal{M}_{EQA}$ the seller randomly chooses between the two query mechanisms above, according to proper probabilities defined in the analysis. We have the following theorem, proved in Appendix C.2.

Mechanism 9 Mechanism $\mathcal{M}_{EQBVCG}$ for Additive Auctions

1: Given $\epsilon > 0$, run algorithm $\mathcal{A}_Q$ with $\delta = (1+\frac{\epsilon}{5})^{1/m} - 1$ and $\epsilon_1 = h(\frac{\epsilon}{10(1+\epsilon)})$ (i.e., $\delta_1 = \frac{\epsilon}{10(1+\epsilon)}$ for Small Tail Assumption 1), for each player i's distribution $\mathcal{D}_{ij}$ on each item j. Denote by $\mathcal{D}'_{ij}$ the returned distribution. Let $\mathcal{D}'_i = \times_{j\in M}\mathcal{D}'_{ij}$ and $\mathcal{D}' = \times_{i\in N}\mathcal{D}'_i$.

2: Run $\mathcal{M}_{BVCG}$ with $\mathcal{D}'$ and the players' reported values, $b = (b_{ij})_{i\in N, j\in M}$, to get allocation $x = (x_{ij})_{i\in N, j\in M}$ and price profile $p = (p_i)_{i\in N}$ as the outcome.

Theorem 7. *$\forall\epsilon > 0$, any additive instance $\mathcal{I} = (N, M, \mathcal{D})$ satisfying Small-Tail Assumption 1, $\mathcal{M}_{EQA}$ is DSIC, has query complexity $O(-m^2 n \log_{1+\frac{\epsilon}{5}} h(\frac{\epsilon}{10(1+\epsilon)})$, and $Rev(\mathcal{M}_{EQA}(\mathcal{I})) \geq \frac{OPT(\mathcal{I})}{8(1+\epsilon)}$.*

As shown in Theorem 7, the query complexity of mechanism $\mathcal{M}_{EQA}$ has an extra factor of m compared with that of $\mathcal{M}_{EVA}$ (and the lower bound). However, the advantages of using quantile queries are not only that we can handle unbounded distributions, but also that we can use the resulting query mechanisms to construct *sampling* mechanisms. See Section 6 for more details.

5.6 Using Quantile Queries for Bounded Distributions

As a corollary, Theorems 5, 6 and 7 also provide another way to approximate the optimal revenue using only quantile queries when the distributions are bounded. More precisely, we have the following, proved in Appendix C.3.

Corollary 1. *For any $\epsilon > 0$, $H > 1$, and prior distribution $\mathcal{D}$ with each $\mathcal{D}_{ij}$ bounded within $[1, H]$, there exist DSIC mechanisms that use $O(mn \log_{1+\epsilon} \frac{nmH(1+\epsilon)}{\epsilon})$ quantile queries for single-item auctions and unit-demand auctions, and use $O(m^2 n \log_{1+\epsilon} \frac{nmH(1+\epsilon)}{\epsilon})$ quantile queries for additive auctions, whose approximation ratios to OPT are respectively $1+\epsilon$, $24(1+\epsilon)$ and $8(1+\epsilon)$.*

6 Applications: Constructive Sampling Mechanisms

Using our techniques for query complexity, we can easily construct sampling mechanisms for multi-parameter auctions. Currently, the sample complexity for unit-demand auctions and bounded distributions has been upper-bounded in [25] via PAC learning approaches, but is not constructive. In this section, we explicitly construct sampling mechanisms for both unit-demand and additive auctions, for arbitrary distributions with small-tails (and for bounded distributions).

The idea is to use samples to approximate *quantile* queries. Mechanism 10 defines our sampling mechanism $\mathcal{M}_{SM}$. Recall that mechanisms $\mathcal{M}_{MRS}$, $\mathcal{M}_{UD}$ and $\mathcal{M}_A$ are known (approximately) optimal DSIC mechanisms for single-item, unit-demand and additive auctions respectively. Note that in mechanism $\mathcal{M}_{SM}$, we use a different method to discretize the quantile space for additive auctions, so as to further reduce its sample complexity. In particular, we have the following theorem, proved in Appendix D.

Mechanism 10 Sampling Mechanism $\mathcal{M}_{SM}$ for Single-Item/Unit-Demand/Additive Auctions

1: For single-item auctions and unit-demand auctions, given $\epsilon > 0$, set $\delta = \frac{\epsilon}{6}$, $\epsilon_1 = h(\frac{2\epsilon}{3(1+\epsilon)})$ and $k = \lceil \log_{1+\delta} \frac{1}{\epsilon_1} \rceil$; define the *quantile vector* as $q = (q_0, q_1, \ldots, q_{k-1}, q_k) = (1, \epsilon_1(1+\delta)^{k-1}, \ldots, \epsilon_1(1+\delta), \epsilon_1)$.
For additive auctions, given $\epsilon > 0$, set $\epsilon_1 = h(\frac{\epsilon}{10(1+\epsilon)})$ and $k = \lfloor \frac{1}{\epsilon_1} \rfloor$; define the *quantile vector* as $q = (q_0, q_1, \ldots, q_{k-1}, q_k) = (1, k\epsilon_1, \ldots, 2\epsilon_1, \epsilon_1)$.

2: For each player i and item j, given t samples $V_{ij}^t = \{v_{ij}^1, \ldots, v_{ij}^t\}$, without loss of generality assume $v_{ij}^1 \geq v_{ij}^2 \geq \cdots \geq v_{ij}^t$. For each quantile q_l, set $v_{ij}^{tq_l}$ to be the value corresponding to the quantile query q_l. (If tq_l is not an integer then the mechanism takes $\lceil tq_l \rceil$.)

3: Construct a discrete distribution $\mathcal{D}'_{ij}$ as follows: $\mathcal{D}'_{ij}(v_{ij}^{tq_l}) = q_l - q_{l+1}$ for each $l \in \{0, \ldots, k-1\}$, and $\mathcal{D}'_{ij}(v_{ij}^{tq_k}) = \epsilon_1$. Finally, let $\mathcal{D}'_i = \times_{j \in M} \mathcal{D}'_{ij}$ for each player i and let $\mathcal{D}' = \times_{i \in N} \mathcal{D}'_i$.

4: Run $\mathcal{M}_{MRS}/\mathcal{M}_{UD}/\mathcal{M}_A$ with distribution $\mathcal{D}'$ and the players' reported values.

Theorem 8. $\forall \epsilon > 0$ *and* $\gamma \in (0,1)$*, for any Bayesian instance* $\mathcal{I} = (N, M, \mathcal{D})$ *satisfying Small-Tail Assumption 2, with* $\tilde{O}(h^{-2}(\frac{2\epsilon}{3(1+\epsilon)}) \cdot (\frac{\epsilon}{1+\epsilon})^{-2})$ *samples, with probability at least* $1-\gamma$*, mechanism* $\mathcal{M}_{SM}$ *achieves revenue at least* $\frac{OPT(\mathcal{I})}{1+\epsilon}$ *for single-item auctions and revenue at least* $\frac{OPT(\mathcal{I})}{24(1+\epsilon)}$ *for unit-demand auctions.*

For any Bayesian instance $\mathcal{I} = (N, M, \mathcal{D})$ *satisfying Small-Tail Assumption 1, with* $\tilde{O}(h^{-2}(\frac{\epsilon}{10(1+\epsilon)})(\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}}))^{-2})$ *samples, with probability at least* $1-\gamma$*, mechanism* $\mathcal{M}_{SM}$ *achieves revenue at least* $\frac{OPT(\mathcal{I})}{8(1+\epsilon)}$ *for additive auctions.*

Remark. Following the convention in the literature, a logarithmic factor depending on γ has been absorbed in $\tilde{O}(\cdot)$. If the values are bounded in $[1, H]$, by defining the tail function h according to H, the resulting sample complexity is $\tilde{O}(m^4 n^2 H^2 (1+\epsilon)^4 \epsilon^{-4})$ for unit-demand auctions and $\tilde{O}(m^4 n^2 H^2 (\frac{1+\epsilon}{\epsilon})^2 (\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}}))^{-2})$ for additive auctions. Note that the upper-bound for unit-demand auctions is weaker than the one in [25], but our mechanisms are constructive. Finally, for bounded distributions and $\epsilon < 1$, combining our techniques and the result of [14], we can construct another sampling mechanism for unit-demand auctions with sample complexity $\tilde{O}(mnH\epsilon^{-3})$.

7 Single-Item Auctions with Regular Distributions

In the literature of Bayesian auction design, regular distributions are an important class and have been widely studied. In this section, we show that when we only consider regular distributions for single-item auctions, the query complexity can be much lower. In fact, we no longer need the small-tail assumptions even when the supports are unbounded. Here our lower- and upper-bounds are *tight* upto a logarithmic factor, and require different techniques from previous sections.

For the lower-bound, recall that in Section 3 we allow the distributions to be irregular. To construct the desired distributions, we can first find the un-queried quantile interval and then move the probability mass from its end points to internal points. Because the distributions can be irregular, we have complete control on where to put the probability mass. However, if the distributions have to be regular then this cannot be done. Instead, we start from two different single-peak revenue curves and construct regular distributions from them. We still want to move probability mass from the end points of the un-queried quantile interval to internal points, but such moves must be continuous in order to preserve regularity.

For the upper-bound, we show that regular distributions satisfy the small-tail property with a properly defined tail function. Thus our results for distributions with small-tails directly apply here.

7.1 Lower Bound

With regular distributions, by [15] it is sufficient to use a single sample to achieve 2-approximation in revenue for *single-player* single-item auctions. Because every distribution is a uniform distribution in the quantile space, a sample for such auctions can be obtained by first choosing a quantile q uniformly at random from $[0, 1]$ and then making a quantile query. Thus, a single query is also sufficient for 2-approximation in this case. As such, unlike Theorem 1 where we have proved lower bounds for the query complexity for arbitrary constant approximations, for regular distributions we consider lower bounds for $(1 + \epsilon)$-approximations, where ϵ is sufficiently small. More precisely, we have the following theorem, which is proved in Appendix E.

Theorem 9. *For any constant $\epsilon \in (0, \frac{1}{64})$, there exists a constant C such that, for any $n \geq 1$, any DSIC Bayesian mechanism $\mathcal{M}$ making less than $Cn\epsilon^{-1}$ non-adaptive value and quantile queries to the oracle, there exists a multi-player single-item Bayesian auction instance $\mathcal{I} = (N, M, \mathcal{D})$ where $|N| = n$ and $\mathcal{D}$ is regular, such that $Rev(\mathcal{M}(\mathcal{I})) < \frac{OPT(\mathcal{I})}{1+\epsilon}$.*

7.2 Upper Bound

Our mechanism $\mathcal{M}_{EMR}$ (i.e., "Efficient quantile Myerson mechanism for Regular distributions") first constructs the distribution $\mathcal{D}'$ that approximates $\mathcal{D}$ using the quantile-query algorithm $\mathcal{A}_Q$ with parameters $\delta = \frac{\epsilon}{4}$ and $\epsilon_1 = \frac{\epsilon^2}{256n}$; and then runs Myerson's mechanism $\mathcal{M}_{MRS}$ on $\mathcal{D}'$. Formally, we have the following theorem, proved in Appendix E.

Theorem 10. $\forall \epsilon \in (0, 1)$*, and for any single-item instance $\mathcal{I} = (N, M, \mathcal{D})$ where $\mathcal{D}$ is regular, mechanism $\mathcal{M}_{EMR}$ is DSIC, has query complexity $O(n \log_{1+\epsilon} \frac{n}{\epsilon})$, and $Rev(\mathcal{M}_{EMR}(\mathcal{I})) \geq \frac{OPT(\mathcal{I})}{1+\epsilon}$.*

Remark. Regular distributions are also well-studied in the sample complexity for single-item auctions. Following [11, 23, 14], the sample complexity in this setting is bounded between $\Omega(\max\{n\epsilon^{-1}, \epsilon^{-3}\})$ and $\tilde{O}(n\epsilon^{-4})$. However, each sample is a valuation profile of the players, and thus contains n values. When ϵ is small, the query complexity in this setting is $O(n\epsilon^{-1} \log \frac{n}{\epsilon})$. Thus the query complexity is still much lower than the sample complexity.

A Missing Materials for Section 3

A.1 Missing Figure

The cumulative probability function of each distribution $\mathcal{D}_z$ is illustrated in Figure 1.

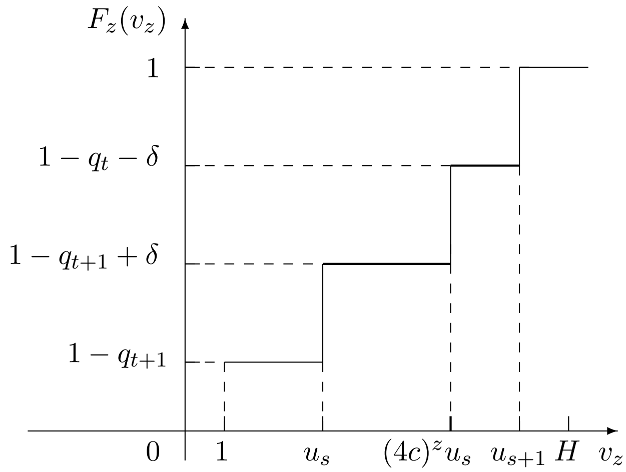

Figure 1: The cumulative probability function of $\mathcal{D}_z$.

A.2 Missing Table

The quantile queries and corresponding answers for the $\mathcal{D}_z$'s are illustrated in Table 3.

Quantile queries	Corresponding values	Oracle's answer
0	$(u_{s+1}, +\infty)$	$+\infty$
$(0, q_t + \delta)$	$\emptyset$	u_{s+1}
$q_t + \delta$	$((4c)^z u_s, u_{s+1}]$	u_{s+1}
$(q_t + \delta, q_{t+1} - \delta)$	$\emptyset$	$(4c)^z u_s$
$q_{t+1} - \delta$	$(u_s, (4c)^z u_s]$	$(4c)^z u_s$
$(q_{t+1} - \delta, q_{t+1})$	$\emptyset$	u_s
q_{t+1}	$(1, u_s]$	u_s
$(q_{t+1}, 1)$	$\emptyset$	1
1	1	1

Table 3: Quantile queries and corresponding answers for $\mathcal{D}_z$.

A.3 Missing Definitions and Proofs

A very broad class of Bayesian auctions, *(monotone) sub-additive* auctions, is such that each player i has a valuation function $v_i : 2^{[m]} \to \mathbb{R}$, which satisfies $v_i(S) + v_i(T) \geq v_i(S \cup T) \geq v_i(S) \geq 0$ for any subsets of items S and T. As such a valuation function in general needs 2^m values to describe, following the conventions in Bayesian auction design [29, 9, 6], we will consider *succinct sub-additive* auctions, where only the item-values, that is, the v_{ij}'s, are independently drawn from the underlying distribution $\mathcal{D} = \times_{i \in [n], j \in [m]} \mathcal{D}_{ij}$. Given $(v_{ij})_{j \in [m]}$, it is publicly known how to compute player i's value for any subset of items. That is, the valuation function v_i now takes a vector of item-values $(v_{ij})_{j\in[m]}$ and a subset $S \subseteq [m]$ as inputs, such that for any vector $(v_{ij})_{j\in[m]}$, the resulting function $v_i((v_{ij})_{j\in[m]}, \cdot)$ is sub-additive and $v_i((v_{ij})_{j\in[m]}, \{j\}) = v_{ij}$ for each item j. Note that such auctions include single-item, unit-demand and additive auctions as special cases.

Theorem 1 (restated)**.** *For any constant $c > 1$, there exists a constant C such that, for any $n \geq 1, m \geq 1$, any large enough H, any succinct sub-additive valuation function profile $v = (v_i)_{i\in[n]}$, and any DSIC Bayesian mechanism $\mathcal{M}$ making less than $Cnm\log_c H$ non-adaptive value and quantile queries to the oracle, there exists a multi-item Bayesian auction instance $\mathcal{I} = (N, M, \mathcal{D})$ with valuation profile v, where $|N| = n, |M| = m$ and the item values are bounded in $[1, H]$, such that $Rev(\mathcal{M}(\mathcal{I})) < \frac{OPT(\mathcal{I})}{c}$.*

Proof. Similar to the proof of Lemma 1, for any H, let $k \triangleq \lfloor \frac{1}{4}\log_{(8c)^{4c+2}} H\rfloor$ and $C \triangleq \frac{1}{24c(4c+2)\log_c(8c)}$. Let H be large enough so that $k \geq 1$. It is easy to see that $Cnm\log_c H < \frac{nmk}{3c}$. Thus, for any DSIC Bayesian mechanism $\mathcal{M}$ that makes less than $Cnm\log_c H$ non-adaptive value and quantile queries, there exists a player-item pair (i^*, j^*), a value interval (u_s, u_{s+1}) and a quantile interval (q_t, q_{t+1}) such that, with probability at least $1-\frac{1}{3c}$, $\mathcal{M}$ does not query these two intervals for player i^*'s value distribution for item j^*. We will focus on i^*, j^* and the two intervals, and show that $\mathcal{M}$ cannot generate good revenue from them.

We construct $\lceil 4c\rceil$ Bayesian instances, $\mathcal{I}^z = (N, M, \mathcal{D}^z)$ with $z \in [\lceil 4c\rceil]$, where each $\mathcal{D}^z = \times_{i\in[n],j\in[m]}\mathcal{D}^z_{ij}$ is the prior distribution for the players' item values. For each z, i, j, the distribution $\mathcal{D}^z_{ij}$ is constantly 1 if $i \neq i^*$ or $j \neq j^*$. Let $\mathcal{D}^z_{i^*j^*}$ be the distribution $\mathcal{D}_z$ defined in Table 2.

Given any succinct sub-additive valuation function profile $v = (v_i)_{i\in[n]}$ where each v_i takes a vector of item-values $(v_{ij})_{j\in[m]}$ as part of its input, we would like to compare the optimal revenue for the sub-additive instances defined by the $\mathcal{I}^z$'s with the corresponding expected revenue of $\mathcal{M}$. By construction, the $\mathcal{D}^z$'s differ only at the $\mathcal{D}^z_{i^*j^*}$'s, within the value interval (u_s, u_{s+1}) and the quantile interval (q_t, q_{t+1}). Accordingly, with probability at least $1 - \frac{1}{3c}$, mechanism $\mathcal{M}$ cannot distinguish the $\mathcal{I}^z$'s from each other. Eventually, we will analyze the revenue of $\mathcal{M}$ conditional on this event happening.

For now, to compare the optimal revenue and that of $\mathcal{M}$, let us first introduce some notations. For any item-value profile $\hat{v} = (\hat{v}_{ij})_{i\in[n],j\in[m]}$, when the players bid $\hat{v}$, we denote by $x_i(\hat{v})$ the (randomized) allocation of $\mathcal{M}$ to a player i. It is defined by the probabilities $\sigma_{iS}(\hat{v})$ for all the subsets $S \subseteq [m]$: each $\sigma_{iS}(\hat{v})$ is the probability that player i receives S under bid $\hat{v}$. Accordingly, the expected value of player i for allocation $x_i(\hat{v})$ is $v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})) = \sum_S v_i((\hat{v}_{ij})_{j\in[m]}, S)\cdot\sigma_{iS}(\hat{v})$. Moreover, for each item j, let $x_{ij}(\hat{v})$ be the probability that player i receives item j according to $x_i(\hat{v})$: that is, $x_{ij}(\hat{v}) = \sum_{S:j\in S}\sigma_{iS}(\hat{v})$.

We upper-bound the revenue of $\mathcal{M}$ in three steps. To begin with, we reduce the multi-player sub-additive instances to *single-player* sub-additive instances, and construct a DSIC Bayesian mechanism $\mathcal{M}^*$ that only sells the items to player i^*. Given any instance $\mathcal{I}^z$, mechanism $\mathcal{M}^*$ runs on the single-player sub-additive instance $\mathcal{I}^z_{i^*} = (\{i^*\}, M, \mathcal{D}^z_{i^*})$. It first simulates the item values of players in $N \setminus \{i^*\}$, which are all 1's, and then runs $\mathcal{M}$. Mechanism $\mathcal{M}^*$ answers the oracle queries of $\mathcal{M}$ truthfully. The allocation and the payment for player i^* under $\mathcal{M}^*$ is the same as those under $\mathcal{M}$. For any player $i \neq i^*$, mechanism $\mathcal{M}^*$ assigns nothing to him and charges him 0, because i is an imaginary player to $\mathcal{M}^*$. It is easy to see that mechanism $\mathcal{M}^*$ is DSIC. Moreover,

$$Rev(\mathcal{M}^*(\mathcal{I}^z_{i^*})) \geq Rev(\mathcal{M}(\mathcal{I}^z)) - \mathbb{E}_{\hat{v}\sim\mathcal{D}^z}\sum_{i\neq i^*} v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})), \tag{6}$$

because the revenue generated by $\mathcal{M}$ from players in $N \setminus \{i^*\}$ is at most their total value for the allocation.

Next, we reduce the single-player sub-additive instances to single-player *additive* instances, and construct a DSIC Bayesian mechanism $\mathcal{M}^+$ that runs on the single-player additive instances $\mathcal{I}_{i^*}^{+z} = (\{i^*\}, M, \mathcal{D}_{i^*}^z)$, with $z \in [\lceil 4c \rceil]$. Note that each $\mathcal{I}_{i^*}^{+z}$ has the same item-value distributions as $\mathcal{I}_{i^*}^z$, but player i^*'s value for any subset of items is additive.

For each single-player sub-additive instance defined by $\mathcal{I}_{i^*}^z$ and the valuation function profile v, by the taxation principle [18], mechanism $\mathcal{M}^*$ is equivalent to providing a menu of options to player i^* and then letting i^* choose a menu entry maximizing his expected utility according to his true valuation. Given any instance $\mathcal{I}_{i^*}^{+z}$, mechanism $\mathcal{M}^+$ provides the same menu as mechanism $\mathcal{M}^*$ under $\mathcal{I}_{i^*}^z$ and v, except that the payment in each entry is discounted by a multiplicative $1-\hat{\epsilon}$. Here $\hat{\epsilon}$ is a sufficiently small constant in $(0,1)$ to be determined later in the analysis. The truthfulness of $\mathcal{M}^+$ is immediate, because it lets i^* choose a menu entry maximizing his expected utility under his true additive values. Let

$$\bar{\delta} \triangleq \mathbb{E}_{\hat{v}_{i^*} \sim \mathcal{D}_{i^*}^z} \max_{S \subseteq [m]} (\sum_{j \in S} \hat{v}_{i^*j} - v_{i^*}((\hat{v}_{i^*j})_{j \in [m]}, S)),$$

the expected maximum difference between the additive values and the succinct sub-additive values. Following Lemma 3.4 in [29], which compares the revenue in the sub-additive instance with that in the corresponding additive instance, we have

$$Rev(\mathcal{M}^+(\mathcal{I}_{i^*}^{+z})) \geq (1-\hat{\epsilon})(Rev(\mathcal{M}^*(\mathcal{I}_{i^*}^z)) - \bar{\delta}/\hat{\epsilon}). \tag{7}$$

Finally, we reduce the single-player additive instances to single-player *single-item* instances, and construct a DSIC Bayesian mechanism $\mathcal{M}'$ that only sells item j^* to player i^*. Mechanism $\mathcal{M}'$ runs on the single-player single-item instances $\mathcal{I}_{i^*j^*}^z = (\{i^*\}, \{j^*\}, \mathcal{D}_{i^*j^*}^z)$, with $z \in [\lceil 4c \rceil]$. Given any $\mathcal{I}_{i^*j^*}^z$, it first lets player i^* report $\hat{v}_{i^*j^*}$. Then it simulates the $\hat{v}_{i^*j}$'s from $\mathcal{D}_{i^*j}^z$ for $j \neq j^*$, which are all 1's, and runs $\mathcal{M}^+$ on the augmented additive instance $\mathcal{I}_{i^*}^{+z}$ to obtain allocation $x_{i^*}^+(\hat{v}_{i^*})$ and payment $p_{i^*}^+(\hat{v}_{i^*})$. For each item j, let $x_{i^*j}^+(\hat{v}_{i^*})$ be the probability that player i^* receives item j in the allocation. Mechanism $\mathcal{M}'$ sets its outcome to be the following:

- $x'_{i^*j^*}(\hat{v}_{i^*j^*}) = x_{i^*j^*}^+(\hat{v}_{i^*})$; and
- $p'_{i^*}(\hat{v}_{i^*j^*}) = p_{i^*}^+(\hat{v}_{i^*}) - \sum_{j \in [m] \setminus \{j^*\}} \hat{v}_{i^*j} x_{i^*j}^+(\hat{v}_{i^*})$.

Note that $p'_{i^*}(\hat{v}_{i^*j^*})$ may be negative. By Lemma 21 of [20], mechanism $\mathcal{M}'$ is DSIC and

$$Rev(\mathcal{M}'(\mathcal{I}_{i^*j^*}^z)) \geq Rev(\mathcal{M}^+(\mathcal{I}_{i^*}^{+z})) - \sum_{j \neq j^*} \mathbb{E}_{\hat{v}_{i^*j} \sim \mathcal{D}_{i^*j}^z} \hat{v}_{i^*j}. \tag{8}$$

Now we combine the above three reduction steps together and consider the event when mechanism $\mathcal{M}$ cannot distinguish the $\mathcal{I}^z$'s from each other. When this happens, mechanism $\mathcal{M}$ produces the same outcome for all the instances. Accordingly, although mechanism $\mathcal{M}^*$ is given the distributions $\mathcal{D}_{i^*}^z$, by simulating $\mathcal{M}$, it still produces the same outcome for all the $\mathcal{I}_{i^*}^z$'s, thus the same menu for all of them. So mechanism $\mathcal{M}^+$ also produces the same menu for all the $\mathcal{I}_{i^*}^{+z}$'s: that is, the menu produced by $\mathcal{M}^*$ with the payments discounted by $1-\hat{\epsilon}$. As a result, although mechanism $\mathcal{M}'$ is given the $\mathcal{D}_{i^*j^*}^z$'s, it still cannot "distinguish" the $\mathcal{I}_{i^*j^*}^z$'s from each other and produces the same outcome for all of them. Following the proof of Lemma 1, in this case there exists $z^* \in [\lceil 4c \rceil]$ such that

$$Rev(\mathcal{M}'(\mathcal{I}_{i^*j^*}^{z^*})) < \frac{1}{2c} OPT(\mathcal{I}_{i^*j^*}^{z^*}).$$

Combining this inequality with Equations 6, 7 and 8, we have

$$
\begin{aligned}
& Rev(\mathcal{M}(\mathcal{I}^{z^*})) \leq Rev(\mathcal{M}^*(\mathcal{I}^{z^*}_{i^*})) + \mathbb{E}_{\hat{v}\sim\mathcal{D}^{z^*}} \sum_{i\neq i^*} v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})) \\
\leq \quad & \frac{Rev(\mathcal{M}^+(\mathcal{I}^{+z^*}_{i^*}))}{1-\hat{\epsilon}} + \bar{\delta}/\hat{\epsilon} + \mathbb{E}_{\hat{v}\sim\mathcal{D}^{z^*}} \sum_{i\neq i^*} v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})) \\
\leq \quad & \frac{1}{1-\hat{\epsilon}} \left(Rev(\mathcal{M}'(\mathcal{I}^{z^*}_{i^*j^*})) + \sum_{j\neq j^*} \mathbb{E}_{\hat{v}_{i^*j}\sim\mathcal{D}^{z^*}_{i^*j}} \hat{v}_{i^*j} \right) + \bar{\delta}/\hat{\epsilon} + \mathbb{E}_{\hat{v}\sim\mathcal{D}^{z^*}} \sum_{i\neq i^*} v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})) \\
< \quad & \frac{1}{1-\hat{\epsilon}} \left(\frac{1}{2c} OPT(\mathcal{I}^{z^*}_{i^*j^*}) + \sum_{j\neq j^*} \mathbb{E}_{\hat{v}_{i^*j}\sim\mathcal{D}^{z^*}_{i^*j}} \hat{v}_{i^*j} \right) + \bar{\delta}/\hat{\epsilon} + \mathbb{E}_{\hat{v}\sim\mathcal{D}^{z^*}} \sum_{i\neq i^*} v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})). \quad (9)
\end{aligned}
$$

Note that $OPT(\mathcal{I}^{z^*}_{i^*j^*}) \leq OPT(\mathcal{I}^{z^*})$, since selling a single item to a single player is a feasible outcome. Moreover, since $\mathcal{D}^{z^*}_{ij}$ is constantly 1 when $i \neq i^*$ or $j \neq j^*$, and since the valuation function profile v is succinct sub-additive, we have

$$
\begin{aligned}
& \sum_{j\neq j^*} \mathbb{E}_{\hat{v}_{i^*j}\sim\mathcal{D}^{z^*}_{i^*j}} \hat{v}_{i^*j} = m-1, \\
& \bar{\delta}/\hat{\epsilon} = \frac{1}{\hat{\epsilon}} \mathbb{E}_{\hat{v}_{i^*}\sim\mathcal{D}^{z^*}_{i^*}} \max_{S\subseteq[m]} (\sum_{j\in S} \hat{v}_{i^*j} - v_{i^*}((\hat{v}_{i^*j})_{j\in[m]}, S)) \leq \frac{m-1}{\hat{\epsilon}}, \text{ and} \\
& \mathbb{E}_{\hat{v}\sim\mathcal{D}^{z^*}} \sum_{i\neq i^*} v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})) \leq m.
\end{aligned}
$$

Here the second equation is because $\sum_{j\in S} \hat{v}_{i^*j} - v_{i^*}((\hat{v}_{i^*j})_{j\in[m]}, S) \leq m-1$ for any $\hat{v}_{i^*}$ and S. The third equation is because $\sum_{i\neq i^*} v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})) \leq m$ for any $\hat{v}$: indeed, each item can be sold to at most one player, generating value 1.

Combining the equations above with Equation 9, we have

$$Rev(\mathcal{M}(\mathcal{I}^{z^*})) < \frac{1}{1-\hat{\epsilon}} \left(\frac{1}{2c} OPT(\mathcal{I}^{z^*}) + m - 1 \right) + \frac{m-1}{\hat{\epsilon}} + m.$$

Setting $\hat{\epsilon} = \frac{1}{4}$, we have

$$Rev(\mathcal{M}(\mathcal{I}^{z^*})) < \frac{2}{3c} OPT(\mathcal{I}^{z^*}) + \frac{19m}{3}. \quad (10)$$

Finally, we combine Equation 10 with the probability that $\mathcal{M}$ cannot distinguish the $\mathcal{I}^z$'s, which is $1 - \frac{1}{3c}$. Recall from the proof of Lemma 1 that $q_0 \geq H^{-\frac{1}{4}}$ and $u_0 \geq H^{\frac{3}{4}}$. By selling item j^* to player i^* at price u_0, we have $OPT(\mathcal{I}^{z^*}) \geq u_0 q_0 \geq \sqrt{H}$. When $H > (\frac{57}{2} mc^2)^2$, $OPT(\mathcal{I}^{z^*}) > \frac{57}{2} mc^2$ and

$$Rev(\mathcal{M}(\mathcal{I}^{z^*})) \leq (1 - \frac{1}{3c})(\frac{2}{3c} OPT(\mathcal{I}^{z^*}) + \frac{19m}{3}) + \frac{1}{3c} OPT(\mathcal{I}^{z^*}) < \frac{1}{c} OPT(\mathcal{I}^{z^*}).$$

This concludes the proof of Theorem 1. □

Note that Theorem 1 applies to *every* succinct sub-additive valuation function profile. Since such auctions contain single-item, unit-demand, and additive auctions as special cases, Theorem 1 automatically applies to those cases.

B Missing Proofs for Section 4

B.1 Unit-Demand Auctions

Before analyzing mechanism $\mathcal{M}_{EVUD}$, let us first recall the *sequential post-price* mechanism $\mathcal{M}_{UD}$. This mechanism processes the players one by one according to an arbitrary order, computes a price for each player i based on remaining items, remaining players and the prior distribution, and lets i choose his utility-maximizing item (or choose none). The revenue of this mechanism is analyzed by reducing the unit-demand instance to the *COPIES* setting, which we introduce below.

For a unit-demand auction instance $\mathcal{I} = (N, M, \mathcal{D})$, the corresponding COPIES instance is denoted by $\mathcal{I}^{CP} = (N^{CP}, M^{CP}, \mathcal{D})$, where each player $i \in N$ has m copies and each item $j \in M$ has n copies, and player i's copy j is only interested in item j's copy i, with value v_{ij} drawn independently from $\mathcal{D}_{ij}$. Thus $N^{CP} = M^{CP} = N \times M$, and $\mathcal{I}^{CP}$ is a single-parameter instance. Denote by N_i the set of player i's copies and by M_j the set of item j's copies. Note that both $\{N_i\}_{i \in N}$ and $\{M_j\}_{j \in M}$ are partitions of N^{CP} (and M^{CP}). Two natural constraints are imposed on feasible allocations under the COPIES setting, so as to connect it with the original unit-demand setting: (1) for each player i, at most one of his copies gets an item; and (2) for each item j, at most one of its copies gets allocated. Accordingly, letting q_s be the probability that a feasible mechanism allocates an item to a player copy $s \in N^{CP}$, we have $\sum_{s \in N_i} q_s \leq 1$ for each $i \in N$ and $\sum_{s \in M_j} q_s \leq 1$ for each $j \in M$.

The corresponding mechanism $\mathcal{M}_{UD}^{CP}$ for the COPIES setting works in the same way as $\mathcal{M}_{UD}$, except that it considers an arbitrary order of the players in N^{CP}, thus different copies of the same player may not be processed together. When evaluating the performance of mechanism $\mathcal{M}_{UD}^{CP}$, the order of the players is chosen by an *online adaptive adversary,* who tries to minimize the expected revenue of the mechanism. Because this adversary is the worst-case for mechanism $\mathcal{M}_{UD}^{CP}$,

$$Rev(\mathcal{M}_{UD}(\mathcal{I}; \mathcal{D}')) \geq Rev(\mathcal{M}_{UD}^{CP}(\mathcal{I}^{CP}; \mathcal{D}'))$$

for any distribution $\mathcal{D}'$, where the latter is the expected revenue of $\mathcal{M}_{UD}^{CP}$ under the online adaptive adversary. Indeed, mechanism $\mathcal{M}_{UD}$ can be considered as $\mathcal{M}_{UD}^{CP}$ under a specific order where all copies of each player come together, thus the revenue is at least that when the order of N^{CP} is adaptively chosen by the adversary. Now we are ready to prove Theorem 3.

Theorem 3 (restated)**.** *$\forall \epsilon > 0$, for any unit-demand Bayesian instance $\mathcal{I} = (N, M, \mathcal{D})$ with values bounded within $[1, H]$, mechanism $\mathcal{M}_{EVUD}$ is DSIC, has query complexity $O(mn \log_{1+\epsilon} H)$, and*

$$Rev(\mathcal{M}_{EVUD}(\mathcal{I})) \geq \frac{1}{24(1+\epsilon)} OPT(\mathcal{I}).$$

Proof. It is easy to see that the query complexity of $\mathcal{M}_{EVUD}$ is $O(mn \log_{1+\epsilon} H)$, since each distribution $\mathcal{D}_{ij}$ needs $O(\log_{1+\epsilon} H)$ value queries. Also, it is immediate that $\mathcal{M}_{EVUD}$ is DSIC.

Below we prove the revenue bound. By construction,

$$Rev(\mathcal{M}_{EVUD}(\mathcal{I})) = Rev(\mathcal{M}_{UD}(\mathcal{I}; \mathcal{D}')). \tag{11}$$

Let $\mathcal{I}' = (N, M, \mathcal{D}')$ and $\mathcal{I}'^{CP} = (N^{CP}, M^{CP}, \mathcal{D}')$. We state the following key lemma, which is proved after the proof of Theorem 3.

Lemma 3. *$Rev(\mathcal{M}_{UD}(\mathcal{I}; \mathcal{D}')) \geq Rev(\mathcal{M}_{UD}^{CP}(\mathcal{I}^{CP}; \mathcal{D}')) = Rev(\mathcal{M}_{UD}^{CP}(\mathcal{I}'^{CP}))$.*

By Theorem 1 of [24], the sequential post-price mechanism is at least a 6-approximation to the optimal BIC revenue in the COPIES setting. Thus

$$Rev(\mathcal{M}^{CP}_{UD}(\mathcal{I}'^{CP})) \geq \frac{1}{6}OPT(\mathcal{I}'^{CP}). \tag{12}$$

Next, because the COPIES setting is a single-parameter setting, and because of the way we discretize the value space in algorithm $\mathcal{A}_V$, by Lemma 5 of [14] we have

$$OPT(\mathcal{I}'^{CP}) \geq \frac{1}{1+\epsilon}OPT(\mathcal{I}^{CP}). \tag{13}$$

Finally, by Theorem 6 of [5], the optimal BIC revenue in the COPIES setting is a 4-approximation to the optimal BIC revenue in the original unit-demand setting. Thus

$$OPT(\mathcal{I}^{CP}) \geq \frac{1}{4}OPT(\mathcal{I}). \tag{14}$$

Combining Equations 11, 12, 13, 14 and Lemma 3, Theorem 3 holds. □

Proof of Lemma 3. The inequality is already explained. Now we prove the equality. For any value profile $v \sim \mathcal{D}$, let v' be v rounded down to the support of $\mathcal{D}'$. That is, for each v_{ij}, v'_{ij} is the largest value in the support of $\mathcal{D}'_{ij}$ that is less than or equal to v_{ij}. Recall that the support of $\mathcal{D}'_{ij}$ is the set $\{v_0, \cdots, v_k\}$ as defined in the query algorithm $\mathcal{A}_V$. By the definition of $\mathcal{D}'_{ij}$, for any $0 \leq l \leq k-1$,

$$\Pr_{v_{ij}\sim\mathcal{D}_{ij}}[v'_{ij} = v_l] = \Pr_{v_{ij}\sim\mathcal{D}_{ij}}[v_{ij} \geq v_l] - \Pr_{v_{ij}\sim\mathcal{D}_{ij}}[v_{ij} \geq v_{l+1}] = q(v_l) - q(v_{l+1}) = q_l - q_{l+1} = \mathcal{D}'_{ij}(v_l),$$

and

$$\Pr_{v_{ij}\sim\mathcal{D}_{ij}}[v'_{ij} = v_k] = \Pr_{v_{ij}\sim\mathcal{D}_{ij}}[v_{ij} \geq v_k] = q(v_k) = q_k = \mathcal{D}'_{ij}(v_k).$$

That is, if v is distributed according to $\mathcal{D}$ then v' is distributed according to $\mathcal{D}'$.

For any value profile v and the corresponding v', arbitrarily fix an order σ of the players in N^{CP}, which is a bijection from $\{1, \cdots, mn\}$ to $\{1, \cdots, mn\}$. Without loss of generality, each player $\sigma(s)$ gets the corresponding item $\sigma(s)$ whenever his true value is greater than or equal to the posted price for him. Below we show that mechanism $\mathcal{M}^{CP}_{UD}$ produces the same outcome no matter the players' true values are v or v'. That is, for any $s \in \{1, \ldots, mn\}$, (1) $\mathcal{M}^{CP}_{UD}$ produces the same price $p_{\sigma(s)}$ under v and v' for player $\sigma(s)$, and (2) $v_{\sigma(s)} \geq p_{\sigma(s)}$ if and only if $v'_{\sigma(s)} \geq p_{\sigma(s)}$.

To prove these two properties, note that by the construction of mechanism $\mathcal{M}^{CP}_{UD}$, the price $p_{\sigma(s)}$ posted to $\sigma(s)$ depends only on the distribution $\mathcal{D}'$ and the set $A_{\sigma(s)}$ of items sold to the players arriving before $\sigma(s)$. Here $p_{\sigma(s)}$ may be randomized if $\mathcal{D}'_{\sigma(s)}$ is irregular, but it always takes value in the support of $\mathcal{D}'_{\sigma(s)}$ (except that, if selling the corresponding item $\sigma(s)$ to player $\sigma(s)$ is not feasible anymore, then $p_{\sigma(s)} = +\infty$).

We prove the two desired properties by induction. When $s = 1$, property (1) trivially holds, because $A_{\sigma(1)} = \emptyset$ under both value profiles. Furthermore, because a realization of $p_{\sigma(1)}$ is always in the support of $\mathcal{D}'_{\sigma(1)}$, and because $v'_{\sigma(1)}$ is $v_{\sigma(1)}$ rounded down to the support of $\mathcal{D}'_{\sigma(1)}$, property (2) holds when $s = 1$.

Now assume (1) and (2) hold for any $s \leq t$ with $t < mn$. We show they also hold for $s = t+1$. Indeed, the inductive hypothesis implies that for any $s \leq t$, $A_{\sigma(s)}$ is the same under the two value profiles. In particular, $A_{\sigma(t+1)}$ is the same, which means the price $p_{\sigma(t+1)}$ is the same. Thus property (1) holds. Property (2) also holds because a realization of $p_{\sigma(t+1)}$ is always in the support of $\mathcal{D}'_{\sigma(t+1)}$.

In sum, for any order σ, mechanism $\mathcal{M}_{UD}^{CP}$ produces the same outcome under the two value profiles v and v', thus the same revenue.

Accordingly, under the online adaptive adversary for $(\mathcal{I}^{CP}; \mathcal{D}')$, the revenue $Rev(\mathcal{M}_{UD}^{CP}(\mathcal{I}^{CP}; \mathcal{D}'))$ is the same as the revenue when the players' true values are obtained by rounding $v \sim \mathcal{D}$ to v'. Because the resulting v' is distributed according to $\mathcal{D}'$, $Rev(\mathcal{M}_{UD}^{CP}(\mathcal{I}^{CP}; \mathcal{D}'))$ is at least the expected revenue of $\mathcal{M}_{UD}^{CP}$ under the online adaptive adversary for $\mathcal{I}'^{CP}$. Indeed, a randomized adversary for $\mathcal{I}'^{CP}$ can simulate the adversary for $(\mathcal{I}^{CP}; \mathcal{D}')$: in each step, given v'_s with $s \in N^{CP}$ being the player in this step, the former first samples v_s from $\mathcal{D}_s$ conditional on v_s rounded down to v'_s, and then uses the latter to decide which player arrives next. Thus,

$$Rev(\mathcal{M}_{UD}^{CP}(\mathcal{I}^{CP}; \mathcal{D}')) \geq Rev(\mathcal{M}_{UD}^{CP}(\mathcal{I}'^{CP})).$$

Similarly,

$$Rev(\mathcal{M}_{UD}^{CP}(\mathcal{I}^{CP}; \mathcal{D}')) \leq Rev(\mathcal{M}_{UD}^{CP}(\mathcal{I}'^{CP})).$$

Therefore $Rev(\mathcal{M}_{UD}^{CP}(\mathcal{I}^{CP}; \mathcal{D}')) = Rev(\mathcal{M}_{UD}^{CP}(\mathcal{I}'^{CP}))$ and Lemma 3 holds. $\square$

B.2 Additive Auctions

Theorem 4 (restated)**.** *$\forall \epsilon > 0$, for any additive instance $\mathcal{I} = (N, M, \mathcal{D})$ with values in $[1, H]$, mechanism $\mathcal{M}_{EVA}$ is DSIC, has query complexity $O(mn \log_{1+\epsilon} H)$, and*

$$Rev(\mathcal{M}_{EVA}(\mathcal{I})) \geq \frac{OPT(\mathcal{I})}{8(1+\epsilon)}.$$

Proof. First, it is easy to see that the query complexity of mechanism $\mathcal{M}_{EVA}$ is $O(mn \log_{1+\delta} H)$, since there are in total mn distributions and each one of them needs $O(\log_{1+\delta} H)$ value queries in the algorithm $\mathcal{A}_V$. Since $\delta = \sqrt{\epsilon+1} - 1$, $O(mn \log_{1+\delta} H) = O(mn \log_{1+\epsilon} H)$. Second, since mechanisms $\mathcal{M}_{BVCG}$ and $\mathcal{M}_{IM}$ are both DSIC, $\mathcal{M}_{EVA}$ is DSIC.

Recall that mechanism $\mathcal{M}_{EVA}$ randomly chooses between running $\mathcal{M}_{EVIM}$ and running $\mathcal{M}_{EVBVCG}$. Therefore, to upper-bound the optimal revenue $OPT(\mathcal{I})$ using $Rev(\mathcal{M}_{EVA}(\mathcal{I}))$, we only need to upper-bound each term in Equation 1 of Section 4.3 using $Rev(\mathcal{M}_{EVIM}(\mathcal{I}))$ and $Rev(\mathcal{M}_{EVBVCG}(\mathcal{I}))$.

Note that when $\mathcal{M}_{EVIM}$ uses the value-query algorithm $\mathcal{A}_V$ to learn a distribution, the parameters are also set to be H and $\delta = \sqrt{\epsilon+1} - 1$. Thus, applying Theorem 2 to each item, we have

$$Rev(\mathcal{M}_{IM}(\mathcal{I})) \leq (1+\delta) Rev(\mathcal{M}_{EVIM}(\mathcal{I})).$$

Following Lemma 13 of [5], although the term Single has changed from its original form, we still have

$$\begin{aligned} \text{Single} &= \sum_i \sum_{v_i \in V_i} \sum_j \mathcal{D}_i(v_i) \cdot \pi_{ij}(v_i) \cdot \tilde{\varphi}_{ij}(v_{ij}) \cdot \Pr_{v_{-i} \sim \mathcal{D}_{-i}} [v_i \in R_j^{(v_{-i})}] \\ &\leq Rev(\mathcal{M}_{IM}(\mathcal{I})) \leq (1+\delta) Rev(\mathcal{M}_{EVIM}(\mathcal{I})). \end{aligned} \tag{15}$$

Next, using Lemmas 14 and 15 of [5], we upper-bound the term Under as follows:

$$\begin{aligned}
\text{Under} &= \sum_i \sum_{v_i \in V_i} \sum_j \mathcal{D}_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i} \in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \cdot v_{ij} \cdot \mathbf{I}_{v_{ij} < (1+\delta)\beta_{ij}(v_{-i})} \\
&= \sum_i \sum_{v_i \in V_i} \sum_j \mathcal{D}_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i} \in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \cdot v_{ij} \\
&\qquad \cdot (\mathbf{I}_{v_{ij} < \beta_{ij}(v_{-i})} + \mathbf{I}_{\beta_{ij}(v_{-i}) \le v_{ij} < (1+\delta)\beta_{ij}(v_{-i})}) \\
&\le \sum_i \sum_{v_i \in V_i} \sum_j \mathcal{D}_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i} \in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \\
&\qquad \cdot (v_{ij} \cdot \mathbf{I}_{v_{ij} < \beta_{ij}(v_{-i})} + (1+\delta)\beta_{ij}(v_{-i}) \cdot \mathbf{I}_{v_{ij} \ge \beta_{ij}(v_{-i})}) \\
&\le Rev(\mathcal{M}_{IM}(\mathcal{I})) + (1+\delta) Rev(\mathcal{M}_{IM}(\mathcal{I})) \le 2(1+\delta)^2 Rev(\mathcal{M}_{EVIM}(\mathcal{I})).
\end{aligned}$$

The second inequality above is by Lemmas 14 and 15 of [5], which respectively upper-bound the term Over and the term Under in the original setting. Indeed, we split our term Under into the sum of the original terms Under and Over. Using the above equation, the approximation ratio to $OPT(\mathcal{I})$ will be $9(1+\epsilon)$ eventually. To get the desired $8(1+\epsilon)$-approximation, we prove a variant of Lemma 15 of [5], which directly upper-bounds our term Under as

$$\text{Under} \le (1+\delta) Rev(\mathcal{M}_{IM}(\mathcal{I})) \le (1+\delta)^2 Rev(\mathcal{M}_{EVIM}(\mathcal{I})). \tag{16}$$

The actual proof of this alternative lemma is tedious and does not provide new insights to our result, thus has been omitted.

Next, we upper-bound the term Over:

$$\begin{aligned}
\text{Over} &= \sum_i \sum_{v_i \in V_i} \sum_j \mathcal{D}_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i} \in V_{-i}} (1+\delta)\beta_{ij}(v_{-i}) \mathcal{D}_{-i}(v_{-i}) \mathbf{I}_{v_{ij} \ge (1+\delta)\beta_{ij}(v_{-i})} \\
&\le (1+\delta) \sum_i \sum_{v_i \in V_i} \sum_j \mathcal{D}_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i} \in V_{-i}} \beta_{ij}(v_{-i}) \mathcal{D}_{-i}(v_{-i}) \mathbf{I}_{v_{ij} \ge \beta_{ij}(v_{-i})} \\
&\le (1+\delta) Rev(\mathcal{M}_{IM}(\mathcal{I})) \le (1+\delta)^2 Rev(\mathcal{M}_{EVIM}(\mathcal{I})). \qquad (17)
\end{aligned}$$

The second inequality above is by Lemma 14 of [5].

Next, we upper-bound the term Tail, which is similar to the analysis of [5], but with the threshold price $\beta_{ij}(v_{-i})$ scaled up by a factor of $(1+\delta)$.

$$\begin{aligned}
\text{Tail} &= \sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \sum_j \sum_{v_{ij}>(1+\delta)\beta_{ij}(v_{-i})+r_i(v_{-i})} \mathcal{D}_{ij}(v_{ij}) \cdot (v_{ij} - (1+\delta)\beta_{ij}(v_{-i})) \\
&\qquad \cdot \Pr_{v_{i,-j}\sim\mathcal{D}_{i,-j}} [\exists k \neq j, v_{ik} - (1+\delta)\beta_{ik}(v_{-i}) \geq v_{ij} - (1+\delta)\beta_{ij}(v_{-i})] \\
&\leq \sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \sum_j \sum_{v_{ij}>(1+\delta)\beta_{ij}(v_{-i})+r_i(v_{-i})} \mathcal{D}_{ij}(v_{ij}) \cdot (v_{ij} - (1+\delta)\beta_{ij}(v_{-i})) \\
&\qquad \cdot \Pr_{v_{i,-j}\sim\mathcal{D}_{i,-j}} [\exists k \neq j, v_{ik} - \beta_{ik}(v_{-i}) \geq v_{ij} - (1+\delta)\beta_{ij}(v_{-i})] \\
&\leq \sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \sum_j \sum_{v_{ij}>(1+\delta)\beta_{ij}(v_{-i})+r_i(v_{-i})} \mathcal{D}_{ij}(v_{ij}) \cdot (v_{ij} - (1+\delta)\beta_{ij}(v_{-i})) \\
&\qquad \cdot \sum_{k=1}^{m} \Pr_{v_{ik}\sim\mathcal{D}_{ik}} [v_{ik} \geq v_{ij} - (1+\delta)\beta_{ij}(v_{-i}) + \beta_{ik}(v_{-i})] \\
&\leq \sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \sum_j \sum_{v_{ij}>(1+\delta)\beta_{ij}(v_{-i})+r_i(v_{-i})} \mathcal{D}_{ij}(v_{ij}) \cdot \sum_{k=1}^{m} (v_{ij} - (1+\delta)\beta_{ij}(v_{-i}) + \beta_{ik}(v_{-i})) \\
&\qquad \cdot \Pr_{v_{ik}\sim\mathcal{D}_{ik}} [v_{ik} \geq v_{ij} - (1+\delta)\beta_{ij}(v_{-i}) + \beta_{ik}(v_{-i})] \\
&\leq \sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \sum_j \sum_{v_{ij}>(1+\delta)\beta_{ij}(v_{-i})+r_i(v_{-i})} \mathcal{D}_{ij}(v_{ij}) \sum_{k=1}^{m} r_{ik}(v_{-i}) \\
&= \sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \sum_j r_i(v_{-i}) \sum_{v_{ij}>(1+\delta)\beta_{ij}(v_{-i})+r_i(v_{-i})} \mathcal{D}_{ij}(v_{ij}) \\
&\leq \sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \sum_j ((1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})) \Pr_{v_{ij}\sim\mathcal{D}_{ij}} [v_{ij} > (1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})] \\
&\leq \sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \sum_j r_{ij}(v_{-i}) = \sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) r_i(v_{-i}) = \sum_i r_i \\
&= r \leq Rev(\mathcal{M}_{IM}(\mathcal{I})) \leq (1+\delta) Rev(\mathcal{M}_{EVIM}(\mathcal{I})). \qquad (18)
\end{aligned}$$

The second inequality above is by union bound. The fourth and sixth inequalities use twice the definition of $r_{ij}(v_{-i})$, which sets the optimal price to maximize the expected revenue generated by selling item j to i. The second equality is by the definition of $r_i(v_{-i})$.

Finally, we upper-bound the term Core. To do so, below we rewrite Core into a different form. Similar to [5], arbitrarily fixing v_{-i} and letting $v_{ij} \sim \mathcal{D}_{ij}$, define the following two new random variables, which again scale the threshold price $\beta_{ij}(v_{-i})$ up by a factor of $(1+\delta)$:

$$b_{ij}(v_{-i}) = (v_{ij} - (1+\delta)\beta_{ij}(v_{-i}))\mathbf{I}_{v_{ij}\geq(1+\delta)\beta_{ij}(v_{-i})},$$

and

$$c_{ij}(v_{-i}) = b_{ij}(v_{-i})\mathbf{I}_{b_{ij}(v_{-i})\leq r_i(v_{-i})}.$$

Therefore, we have

$$\text{Core} = \sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \sum_j \mathbb{E}_{v_{ij}\sim\mathcal{D}_{ij}}[c_{ij}(v_{-i})].$$

Letting $e_i(v_{-i}) = \sum_j \mathbb{E}_{v_{ij}\sim\mathcal{D}_{ij}}[c_{ij}(v_{-i})] - 2r_i(v_{-i})$, following the proof of Lemma 12 in [5], we still have

$$\Pr[\sum_j b_{ij}(v_{-i}) \geq e_i(v_{-i})] \geq \frac{1}{2}.$$

In the following, we use the revenue of mechanisms $\mathcal{M}_{EVBVCG}$ and $\mathcal{M}_{EVIM}$ to bound the Core. To do so, first note that by the construction of mechanism $\mathcal{M}_{EVBVCG}$,

$$Rev(\mathcal{M}_{EVBVCG}(\mathcal{I})) = Rev(\mathcal{M}_{BVCG}(\mathcal{I};\mathcal{D}')).$$

Let V'_{ij} be the support of $\mathcal{D}'_{ij}$, $V'_i = \times_{j\in M} V'_{ij}$, $V' = \times_{i\in N} V'_i$. As before, given $v_i \sim \mathcal{D}_i$, denote by $v'_i \in V'_i$ the value vector obtained by rounding v_i down to the support of $\mathcal{D}'_i$. That is, each v'_{ij} is the largest value in V'_{ij} that is less than or equal to v_{ij}. Then,

$$\begin{aligned} Rev(\mathcal{M}_{BVCG}(\mathcal{I};\mathcal{D}')) &\geq \sum_i \mathbb{E}_{v_{-i}\sim\mathcal{D}_{-i}} \mathbb{E}_{v_i\sim\mathcal{D}_i} Rev(\mathcal{M}_{BVCG}(v'_i, v_{-i};\mathcal{D}')) \\ &= \sum_i \mathbb{E}_{v_{-i}\sim\mathcal{D}_{-i}} \mathbb{E}_{v'_i\sim\mathcal{D}'_i} Rev(\mathcal{M}_{BVCG}(v'_i, v_{-i};\mathcal{D}')). \end{aligned}$$

The inequality is because each player i can potentially buy item j only when j is in his winning set (i.e., he is the highest bidder for j), and i's winning set under v'_i is a subset of his wining set under v_i. Moreover, the entry fee of i is the same under both (v_i, v_{-i}) and (v'_i, v_{-i}), as it only depends on $\mathcal{D}'_i$ and v_{-i}. Thus the revenue inside the expectation does not increase when v_i is replaced by v'_i. The equality is again because drawing v_i from $\mathcal{D}_i$ and then rounding down to v'_i is equivalent to drawing v'_i from $\mathcal{D}'_i$ directly.

Next, we lower-bound $\sum_i \mathbb{E}_{v_{-i}\sim\mathcal{D}_{-i}} \mathbb{E}_{v'_i\sim\mathcal{D}'_i} Rev(\mathcal{M}_{BVCG}(v'_i, v_{-i};\mathcal{D}'))$. As before, arbitrarily fixing v_{-i} and letting $v'_{ij} \sim \mathcal{D}'_{ij}$, define

$$b'_{ij}(v_{-i}) = (v'_{ij} - \beta_{ij}(v_{-i}))\mathbf{I}_{v'_{ij}\geq\beta_{ij}(v_{-i})}.$$

Note that $b'_{ij}(v_{-i})$ is a random variable that represents player i's utility in the second price mechanism on item j with value $v'_{ij} \sim \mathcal{D}'_{ij}$, when the other players' bids are $v_{-i,j}$. Also note that $\mathcal{M}_{BVCG}$ uses the optimal entry fee for each i with respect to v_{-i} and $\mathcal{D}'$, which generates expected revenue from i (over $\mathcal{D}'_i$) greater than or equal to that by using the following entry fee,

$$e'_i(v_{-i}) = \frac{e_i(v_{-i})}{1+\delta}.$$

Now we show player i accepts the entry fee $e'_i(v_{-i})$ with probability at least $\frac{1}{2}$. Indeed, for any v_i and the corresponding v'_i,

$$\begin{aligned} \sum_j b'_{ij}(v_{-i}) &= \sum_j (v'_{ij} - \beta_{ij}(v_{-i}))\mathbf{I}_{v'_{ij}\geq\beta_{ij}(v_{-i})} \geq \sum_j (\frac{v_{ij}}{1+\delta} - \beta_{ij}(v_{-i}))\mathbf{I}_{\frac{v_{ij}}{1+\delta}\geq\beta_{ij}(v_{-i})} \\ &= \frac{1}{1+\delta}\sum_j (v_{ij} - (1+\delta)\beta_{ij}(v_{-i}))\mathbf{I}_{v_{ij}\geq(1+\delta)\beta_{ij}(v_{-i})} = \frac{1}{1+\delta}\sum_j b_{ij}(v_{-i}). \end{aligned}$$

The inequality is because $v'_{ij} \geq \frac{v_{ij}}{1+\delta}$, and because $\frac{v_{ij}}{1+\delta} \geq \beta_{ij}(v_{-i})$ implies $v'_{ij} \geq \beta_{ij}(v_{-i})$. Therefore

$$\Pr_{v'_i\sim\mathcal{D}'_i}[\sum_j b'_{ij}(v_{-i}) \geq e'_i(v_{-i})] \geq \Pr_{v_i\sim\mathcal{D}_i}[\frac{1}{1+\delta}\sum_j b_{ij}(v_{-i}) \geq \frac{e_i(v_{-i})}{1+\delta}] = \Pr_{v_i\sim\mathcal{D}_i}[\sum_j b_{ij}(v_{-i}) \geq e_i(v_{-i})] \geq \frac{1}{2},$$

as desired. Thus we have

$$\begin{aligned}
Rev(\mathcal{M}_{EVBVCG}(\mathcal{I})) &\geq \sum_i \mathbb{E}_{v_{-i}\sim\mathcal{D}_{-i}} \mathbb{E}_{v'_i\sim\mathcal{D}'_i} Rev(\mathcal{M}_{BVCG}(v'_i, v_{-i}; \mathcal{D}')) \\
&\geq \frac{1}{2}\sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \cdot \frac{e_i(v_{-i})}{1+\delta} \\
&= \frac{1}{2(1+\delta)}\sum_i \sum_{v_{-i}\in V_{-i}} \mathcal{D}_{-i}(v_{-i}) \left(\sum_j \mathbb{E}_{v_{ij}\sim\mathcal{D}_{ij}}[c_{ij}(v_{-i})] - 2r_i(v_{-i})\right) \\
&= \frac{1}{2(1+\delta)}\text{Core} - \frac{r}{1+\delta}.
\end{aligned}$$

That is,

$$\text{Core} \leq 2(1+\delta)Rev(\mathcal{M}_{EVBVCG}(\mathcal{I})) + 2r \leq 2(1+\delta)\left[Rev(\mathcal{M}_{EVBVCG}(\mathcal{I})) + Rev(\mathcal{M}_{EVIM}(\mathcal{I}))\right]. \quad (19)$$

Combining Inequalities 1, 15, 16, 17, 18 and 19,

$$\begin{aligned}
OPT(\mathcal{I}) &\leq (1+\delta)^2 \left(2Rev(\mathcal{M}_{EVBVCG}(\mathcal{I})) + 6Rev(\mathcal{M}_{EVIM}(\mathcal{I}))\right) \\
&= (1+\epsilon)\left(2Rev(\mathcal{M}_{EVBVCG}(\mathcal{I})) + 6Rev(\mathcal{M}_{EVIM}(\mathcal{I}))\right).
\end{aligned}$$

Accordingly, by running mechanism $\mathcal{M}_{EVBVCG}$ with probability $\frac{1}{4}$ and mechanism $\mathcal{M}_{EVIM}$ with probability $\frac{3}{4}$, the expected revenue of mechanism $\mathcal{M}_{EVA}$ is

$$Rev(\mathcal{M}_{EVA}(\mathcal{I})) \geq \frac{1}{8(1+\epsilon)} OPT(\mathcal{I}).$$

This finishes the proof of Theorem 4. □

C Missing Proofs for Section 5

C.1 Single-Item Auctions

Claim 1 (restated). *$\mathcal{M}^*$ is DSIC.*

Proof. Because $\mathcal{M}_{MRS}$ is DSIC, each x_i is monotone in v_i. Although v_i is a random variable given v'_i, it is easy to see that for any two different values $v'_i \in V'_i$ and $\hat{v}'_i \in V'_i$, the corresponding resampled values v_i and $\hat{v}_i$ are such that $v'_i < \hat{v}'_i$ implies $v_i \leq \hat{v}_i$. Thus x_i is monotone in v'_i as well. Moreover, let θ_i be player i's threshold payment in $\mathcal{M}_{MRS}$ given v_{-i} and $\mathcal{D}$. If $v'_i > \theta_i$ then $v_i > \theta_i$, thus player i gets the item at price $p_i = \theta_i$. If $v'_i < \theta_i$ then player i does not get the item and $p_i = 0$, no matter whether $v_i < \theta_i$ or not. Accordingly, θ_i is also player i's threshold payment in $\mathcal{M}^*$ under v_{-i} and $\mathcal{D}'$. Since v_{-i} does not depend on v'_i, $\mathcal{M}^*$ is DSIC as desired. □

Claim 2 (restated). $\Pr_{v_i\sim\mathcal{D}_i}[v_i \geq p_i(v_{-i};\mathcal{D}) | q_i(v_i) > \epsilon_1] \leq (1+\delta)\Pr_{v_i\sim\mathcal{D}_i}[v_i^- \geq p_i(v_{-i};\mathcal{D})]$.

Proof. By definition, $q_i(v_i) > \epsilon_1$ implies $v_i \leq v'_{i;k}$, where $v'_{i;k}$ is the largest value in V'_i, the support of distribution $\mathcal{D}'_i$. Note that $v_i^- \leq v'_{i;k}$ for any v_i. If $p_i(v_{-i};\mathcal{D}) > v'_{i;k}$, then both probabilities are 0 and the inequality holds.

Below we consider the case $p_i(v_{-i};\mathcal{D}) \leq v'_{i;k}$. Let $v'_{i;-1} = -1$ and $l \in \{0, 1, \ldots, k\}$ be such that $v'_{i;l} \geq p_i(v_{-i};\mathcal{D})$ and $v'_{i;l-1} < p_i(v_{-i};\mathcal{D})$. We have

$$
\begin{aligned}
& \Pr_{v_i \sim \mathcal{D}_i}[v_i \geq p_i(v_{-i}; \mathcal{D}) | q_i(v_i) > \epsilon_1] \\
\leq \; & \Pr_{v_i \sim \mathcal{D}_i}[v_i^+ \geq p_i(v_{-i}; \mathcal{D}) | q_i(v_i) > \epsilon_1] \\
= \; & \Pr_{v_i \sim \mathcal{D}_i}[v_i^+ \geq v'_{i;l} | q_i(v_i) > \epsilon_1] \\
= \; & \Pr_{v_i \sim \mathcal{D}_i}[v_i^- \geq v'_{i;l-1} | q_i(v_i) > \epsilon_1] \\
= \; & \Pr_{v_i \sim \mathcal{D}_i}[v_i^- \geq v'_{i;\max\{0,l-1\}} | q_i(v_i) > \epsilon_1] \\
= \; & \frac{\Pr_{v_i \sim \mathcal{D}_i}[v_i^- \geq v'_{i;\max\{0,l-1\}} \text{ and } q_i(v_i) > \epsilon_1]}{\Pr_{v_i \sim \mathcal{D}_i}[q_i(v_i) > \epsilon_1]} \\
= \; & \frac{\Pr_{v_i \sim \mathcal{D}_i}[v_i^- \geq v'_{i;\max\{0,l-1\}}] - \Pr_{v_i \sim \mathcal{D}_i}[v_i^- \geq v'_{i;\max\{0,l-1\}} \text{ and } q_i(v_i) \leq \epsilon_1]}{\Pr_{v_i \sim \mathcal{D}_i}[q_i(v_i) > \epsilon_1]} \\
= \; & \frac{\Pr_{v_i \sim \mathcal{D}_i}[v_i^- \geq v'_{i;\max\{0,l-1\}}] - \Pr_{v_i \sim \mathcal{D}_i}[q_i(v_i) \leq \epsilon_1]}{\Pr_{v_i \sim \mathcal{D}_i}[q_i(v_i) > \epsilon_1]} \\
\leq \; & \Pr_{v_i \sim \mathcal{D}_i}[v_i^- \geq v'_{i;\max\{0,l-1\}}] \\
= \; & \Pr_{v'_i \sim \mathcal{D}'_i}[v'_i \geq v'_{i;\max\{0,l-1\}}] = q_{\max\{0,l-1\}} \leq (1+\delta) q_l \\
= \; & (1+\delta) \Pr_{v'_i \sim \mathcal{D}'_i}[v'_i \geq v'_{i;l}] = (1+\delta) \Pr_{v'_i \sim \mathcal{D}'_i}[v'_i \geq p_i(v_{-i}; \mathcal{D})] \\
= \; & (1+\delta) \Pr_{v_i \sim \mathcal{D}_i}[v_i^- \geq p_i(v_{-i}; \mathcal{D})],
\end{aligned}
$$

as desired. Indeed, the first inequality is because $v_i^+ > v_i$, and the first equality is because $v_i^+ \in V'_i \cup \{+\infty\}$ and thus $v_i^+ \geq p_i(v_{-i}; \mathcal{D})$ if and only if $v_i^+ \geq v'_{i;l}$. Similarly, the second equality is because (v_i^-, v_i^+) and $(v'_{i;l-1}, v'_{i;l})$ are two pairs of consecutive values in $V'_i \cup \{-1, +\infty\}$, thus $v_i^+ \geq v'_{i;l}$ if and only if $v_i^- \geq v'_{i;l-1}$. The third equality is because $v_i^- \geq v'_{i;0}$ always. The sixth equality is because $q_i(v_i) \leq \epsilon_1$ implies $v_i \geq v'_{i;k} \geq v'_{i;l}$, thus $v_i^- \geq v'_{i;\max\{0,l-1\}}$. The seventh equality is by the definition of the round-down scheme. The following two equalities and the inequality are by the construction of $\mathcal{D}'_i$ and the definition of the quantile vector q. Indeed, $(1+\delta)q_0 = 1+\delta > 1 = q_0$, $(1+\delta)q_1 = \epsilon_1(1+\delta)^k \geq \epsilon_1(1+\delta)^{\log_{1+\delta} \frac{1}{\epsilon_1}} = \epsilon_1 \cdot \frac{1}{\epsilon_1} = 1 = q_0$, and $(1+\delta)q_l = q_{l-1}$ for any $l \geq 2$. The second-last equality is because $v'_i \in V'_i$, thus $v'_i \geq v'_{i;l}$ if and only if $v'_i \geq p_i(v_{-i}; \mathcal{D})$. Finally, the last equality is again by the definition of the round-down scheme. $\square$

Claim 3 (restated)**.** *$Rev(\mathcal{M}^*(\mathcal{I}')) \geq \frac{1}{1+\epsilon} OPT(\mathcal{I})$.*

Proof. Combining Equation 5 and Claim 2, we have

$$
Rev(\mathcal{M}^*(\mathcal{I}')) \geq \frac{1}{1+\delta} \sum_i \mathop{\mathbb{E}}_{v_{-i} \sim \mathcal{D}_{-i}} p_i(v_{-i}; \mathcal{D}) \cdot \Pr_{v_i \sim \mathcal{D}_i}[v_i \geq p_i(v_{-i}; \mathcal{D}) | q_i(v_i) > \epsilon_1].
$$

Accordingly,

$$
\begin{aligned}
Rev(\mathcal{M}^*(\mathcal{I}')) &\geq \frac{1}{1+\delta}\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} p_i(v_{-i};\mathcal{D})\cdot \Pr_{v_i\sim\mathcal{D}_i}[v_i \geq p_i(v_{-i};\mathcal{D}) | q_i(v_i) > \epsilon_1] \\
\geq & \frac{1}{1+\delta}\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} p_i(v_{-i};\mathcal{D})\cdot \Pr_{v_i\sim\mathcal{D}_i}[q_i(v_i) > \epsilon_1 \text{ and } v_i \geq p_i(v_{-i};\mathcal{D})] \\
= & \frac{1}{1+\delta}\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} p_i(v_{-i};\mathcal{D})\cdot \mathbf{I}_{q_i(v_i)>\epsilon_1}\cdot \mathbf{I}_{v_i\geq p_i(v_{-i};\mathcal{D})} \\
= & \frac{1}{1+\delta}\mathop{\mathbb{E}}_{v\sim\mathcal{D}} \sum_i p_i(v_{-i};\mathcal{D})\cdot \mathbf{I}_{q_i(v_i)>\epsilon_1}\cdot \mathbf{I}_{v_i\geq p_i(v_{-i};\mathcal{D})} \\
\geq & \frac{1}{1+\delta}\mathop{\mathbb{E}}_{v\sim\mathcal{D}} \mathbf{I}_{\forall i, q_i(v_i)>\epsilon_1}\cdot \sum_i p_i(v_{-i};\mathcal{D})\mathbf{I}_{v_i\geq p_i(v_{-i};\mathcal{D})} \\
= & \frac{1}{1+\delta}\mathop{\mathbb{E}}_{v\sim\mathcal{D}} \mathbf{I}_{\forall i, q_i(v_i)>\epsilon_1}\cdot Rev_{OPT}(v;\mathcal{I}) \\
\geq & \frac{1-\delta_1}{1+\delta} OPT(\mathcal{I}).
\end{aligned} \tag{20}
$$

Here the second last equality holds by the definition of $p_i(v_{-i};\mathcal{D})$ and $Rev_{OPT}(v;\mathcal{I})$, and last inequality holds by the Small-Tail Assumption 2. Since $\delta = \frac{\epsilon}{3}$ and $\delta_1 = \frac{2\epsilon}{3(1+\epsilon)}$, we have

$$\frac{1-\delta_1}{1+\delta} = \frac{1}{1+\epsilon},$$

thus Claim 3 holds. □

C.2 Additive Auctions

Before proving Theorem 7, we first analyze mechanism $\mathcal{M}_{EQBVCG}$, and we have the following.

Lemma 4. *$\forall\epsilon > 0$, for any additive Bayesian instance $\mathcal{I} = (N, M, \mathcal{D})$ satisfying Small-Tail Assumption 1, $\mathcal{M}_{EQBVCG}$ is DSIC, has query complexity $O(-m^2 n \log_{1+\frac{\epsilon}{5}} h(\frac{\epsilon}{10(1+\epsilon)}))$, and*

$$Rev(\mathcal{M}_{EQBVCG}(\mathcal{I})) \geq \frac{1}{1+\frac{\epsilon}{5}}\left(Rev(\mathcal{M}_{BVCG}(\mathcal{I})) - \frac{\epsilon}{10(1+\epsilon)}OPT(\mathcal{I})\right).$$

Proof. First, mechanism $\mathcal{M}_{EQBVCG}$ is DSIC because $\mathcal{M}_{BVCG}$ is DSIC. The query complexity is also immediate.

We now focus on the revenue of this mechanism. We explicitly write $\mathcal{M}_{BVCG}(\mathcal{I};\mathcal{D}')$ to emphasize the fact that the seller runs mechanism $\mathcal{M}_{BVCG}$ on the true valuation profile $v\sim\mathcal{D}$, but uses $\mathcal{D}'$ to compute the entry fees e_i. Given a player i and a valuation profile v, $p_i(v_i, \mathcal{D}_i, v_{-i})$ is the price for i under $\mathcal{D}_i$: that is, $p_i(v_i, \mathcal{D}_i, v_{-i}) = \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(\mathcal{D}_i, v_{-i})}(e(\mathcal{D}_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}})$, where we omit v_{-i} in $\beta_{ij}(v_{-i})$ when v_{-i} is clear from the context. [2] The price $p_i(v_i, \mathcal{D}'_i, v_{-i})$ is similarly defined. By the definition of the mechanism, we have

$$Rev(\mathcal{M}_{EQBVCG}(\mathcal{I})) = Rev(\mathcal{M}_{BVCG}(\mathcal{I};\mathcal{D}')) = \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} p_i(v_i, \mathcal{D}'_i, v_{-i}). \tag{21}$$

[2] If there are ties in the players' values, then we distinguish between β^+_{ij} and β^-_{ij}, depending on the identity of the player with the highest bid for j in $N\setminus\{i\}$.

Next, let V'_{ij} be the support of $\mathcal{D}'_{ij}$, $V'_i = \times_{j\in M} V'_{ij}$, round v_i down to the closest valuation v'_i in V'_i and compare the two valuation profiles (v'_i, v_{-i}) and (v_i, v_{-i}). By definition, $v'_{ij} \geq \beta_{ij}$ implies $v_{ij} \geq \beta_{ij}$. Moreover, the entry fee of i is the same under both valuation profiles, as it only depends on $\mathcal{D}'_i$ and v_{-i}. Similarly, the reserve price β_{ij} is the same for any item j. Thus we have $e(\mathcal{D}'_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}} \geq e(\mathcal{D}'_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v'_{ij}\geq\beta_{ij}}$ and $\mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} \geq \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v'_{ij}-\beta_{ij})\geq e(\mathcal{D}'_i,v_{-i})}$. Therefore

$$\mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} p_i(v_i, \mathcal{D}'_i, v_{-i}) \geq \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} p_i(v'_i, \mathcal{D}'_i, v_{-i}) = \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v'_i\sim\mathcal{D}'_i} p_i(v'_i, \mathcal{D}'_i, v_{-i}), \tag{22}$$

where the equality is again because drawing v_i from $\mathcal{D}_i$ and then rounding down to v'_i is equivalent to drawing v'_i from $\mathcal{D}'_i$ directly.

In mechanism $\mathcal{M}_{BVCG}$, given v_{-i} and $\mathcal{D}'_i$, $e(\mathcal{D}'_i, v_{-i})$ is the optimal entry fee for maximizing the expected revenue generated from i, where the expectation is taken over $\mathcal{D}'_i$. Accordingly,

$$\mathop{\mathbb{E}}_{v'_i\sim\mathcal{D}'_i} p_i(v'_i, \mathcal{D}'_i, v_{-i}) \geq \mathop{\mathbb{E}}_{v'_i\sim\mathcal{D}'_i} p_i(v'_i, \mathcal{D}_i, v_{-i}). \tag{23}$$

Combining Equations 21, 22 and 23, we have

$$Rev(\mathcal{M}_{EQBVCG}(\mathcal{I})) \geq \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v'_i\sim\mathcal{D}'_i} p_i(v'_i, \mathcal{D}_i, v_{-i}). \tag{24}$$

Thus we will use $\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v'_i\sim\mathcal{D}'_i} p_i(v'_i, \mathcal{D}_i, v_{-i})$ to upper-bound $Rev(\mathcal{M}_{BVCG}(\mathcal{I}))$.

To do so, first, for any player i, item j and value v_{ij}, if $v_{ij} < v'_{ij;k}$ where $v'_{ij;k}$ is the largest value in V'_{ij}, then denote by $\overline{v}_{ij}$ the smallest value in V'_{ij} that is strictly larger than v_{ij}; otherwise, let $\overline{v}_{ij} = v_{ij}$. Moreover, denote by $\underline{v}_{ij}$ the largest value in V'_{ij} that is weakly smaller than v_{ij}. The valuation $\overline{v}_i$ and $\underline{v}_i$ are defined correspondingly given v_i. Then We have

$$\begin{aligned}
&Rev(\mathcal{M}_{BVCG}(\mathcal{I})) = \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} p_i(v_i, \mathcal{D}_i, v_{-i}) \\
= &\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} \left(e(\mathcal{D}_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}} \right) \\
= &\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} \mathbf{I}_{\forall j, q_{ij}(v_{ij})>\epsilon_1} \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} \left(e(\mathcal{D}_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}} \right) \\
+ &\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} \mathbf{I}_{\exists j, q_{ij}(v_{ij})\leq\epsilon_1} \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} \left(e(\mathcal{D}_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}} \right).
\end{aligned} \tag{25}$$

Below we upper-bound the last two lines in Equation 25 separately. For the first part, we have

$$\begin{aligned}
&\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} \mathbf{I}_{\forall j, q_{ij}(v_{ij})>\epsilon_1} \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} \left(e(\mathcal{D}_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}} \right) \\
\leq &\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} \mathbf{I}_{\forall j, q_{ij}(v_{ij})>\epsilon_1} \mathbf{I}_{\sum_{j:\overline{v}_{ij}\geq\beta_{ij}}(\overline{v}_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} \left(e(\mathcal{D}_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{\overline{v}_{ij}\geq\beta_{ij}} \right) \\
= &\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \sum_{u_i\in V'_i:\sum_{j:u_{ij}\geq\beta_{ij}}(u_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} \Pr_{v_i\sim\mathcal{D}_i}[\overline{v}_i = u_i] \left(e(\mathcal{D}_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{u_{ij}\geq\beta_{ij}} \right).
\end{aligned} \tag{26}$$

The inequality above is because $v_{ij} \leq \overline{v}_{ij}$ for each player i and item j, which implies $\mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} \leq \mathbf{I}_{\sum_{j:\overline{v}_{ij}\geq\beta_{ij}}(\overline{v}_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})}$ and $\sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}} \leq \sum_j \beta_{ij}\mathbf{I}_{\overline{v}_{ij}\geq\beta_{ij}}$. Next, by the definition of the quantile vector q, for any $u_{ij} \in V'_{ij}$ we have

$$\Pr_{v_{ij}\sim\mathcal{D}_{ij}}[\overline{v}_{ij} = u_{ij}] \leq (1+\delta) \Pr_{v_{ij}\sim\mathcal{D}_{ij}}[\underline{v}_{ij} = u_{ij}].$$

Indeed, when $u_{ij} = v'_{ij;0}$, $\Pr[v_{ij} < u_{ij}] = 0 < (1+\delta)(1-\epsilon_1(1+\delta)^{k-1}) = (1+\delta)(q_0 - q_1) = (1+\delta)\Pr[v_{ij} \in [v'_{ij;0}, v'_{ij;1})]$. When $u_{ij} = v'_{ij;l}$ with $0 < l < k$, $\Pr(v_{ij} \in [v'_{ij;l-1}, v'_{ij;l})) = q_{l-1} - q_l \leq (1+\delta)q_l - q_l = \delta q_l = (1+\delta)\delta q_{l+1} = (1+\delta)((1+\delta)q_{l+1} - q_{l+1}) = (1+\delta)(q_l - q_{l+1}) = (1+\delta)\Pr[v_{ij} \in [v'_{ij;l}, v'_{ij;l+1})]$. And when $u_{ij} = v'_{ij;k}$, $\Pr[v_{ij} \in [v'_{ij;k-1}, v'_{ij;k}]] = q_{k-1} - q_k = \delta\epsilon_1 < \epsilon_1 = \Pr[v_{ij} \geq v'_{ij;k}]$. Since all distributions are independent, for any $u_i \in V'_i$ we have

$$\Pr_{v_i\sim\mathcal{D}_i}[\overline{v}_i = u_i] \leq (1+\delta)^m \Pr_{v_i\sim\mathcal{D}_i}[\underline{v}_i = u_i]. \tag{27}$$

Combining Equations 26 and 27, we have

$$\begin{aligned}
& \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} \mathbf{I}_{\forall j, q_{ij}(v_{ij})>\epsilon_1} \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} \left(e(\mathcal{D}_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}} \right) \\
\leq \; & \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \sum_{u_i\in V'_i:\, \sum_{j:u_{ij}\geq\beta_{ij}}(u_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} (1+\delta)^m \cdot \Pr_{v_i\sim\mathcal{D}_i}[\underline{v}_i = u_i] \cdot \left(e(\mathcal{D}_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{u_{ij}\geq\beta_{ij}} \right) \\
= \; & (1+\delta)^m \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v'_i\sim\mathcal{D}'_i} \mathbf{I}_{\sum_{j:v'_{ij}\geq\beta_{ij}}(v'_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} \left(e(\mathcal{D}_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v'_{ij}\geq\beta_{ij}} \right) \\
= \; & (1+\delta)^m \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v'_i\sim\mathcal{D}'_i} p_i(v'_i, \mathcal{D}_i, v_{-i}) \leq (1+\delta)^m Rev(\mathcal{M}_{ECBVCG}(\hat{\mathcal{I}})).
\end{aligned} \tag{28}$$

The first equality above holds because drawing v_i from $\mathcal{D}_i$ and rounding down to the support of $\mathcal{D}'_i$ is equivalent to drawing v'_i from $\mathcal{D}'_i$. The second equality is by the definition of $p_i(v'_i, \mathcal{D}_i, v_{-i})$, and the last inequality holds by Equation 24.

By Equations 25 and 28, we have

$$\begin{aligned}
& Rev(\mathcal{M}_{BVCG}(\mathcal{I})) \\
\leq \; & (1+\delta)^m Rev(\mathcal{M}_{EQBVCG}(\mathcal{I})) \\
+ \; & \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} \mathbf{I}_{\exists j, q_{ij}(v_{ij})\leq\epsilon_1} \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(\mathcal{D}_i,v_{-i})} \left(e(\mathcal{D}_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}} \right).
\end{aligned} \tag{29}$$

For the last line of Equation 29, we have

$$\begin{aligned}
&\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}_{-i}} \mathop{\mathbb{E}}_{v_i\sim\mathcal{D}_i} \mathbf{I}_{\exists j, q_{ij}(v_{ij})\le\epsilon_1} \mathbf{I}_{\sum_{j:v_{ij}\ge\beta_{ij}}(v_{ij}-\beta_{ij})\ge e(\mathcal{D}_i,v_{-i})} \left(e(\mathcal{D}_i,v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\ge\beta_{ij}}\right) \\
= &\mathop{\mathbb{E}}_{v\sim\mathcal{D}} \sum_i \mathbf{I}_{\exists j, q_{ij}(v_{ij})\le\epsilon_1} \mathbf{I}_{\sum_{j:v_{ij}\ge\beta_{ij}}(v_{ij}-\beta_{ij})\ge e(\mathcal{D}_i,v_{-i})} \left(e(\mathcal{D}_i,v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\ge\beta_{ij}}\right) \\
\le &\mathop{\mathbb{E}}_{v\sim\mathcal{D}} \mathbf{I}_{\exists i,j, q_{ij}(v_{ij})\le\epsilon_1} \sum_i \mathbf{I}_{\sum_{j:v_{ij}\ge\beta_{ij}}(v_{ij}-\beta_{ij})\ge e(\mathcal{D}_i,v_{-i})} \left(e(\mathcal{D}_i,v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\ge\beta_{ij}}\right) \\
= &\mathop{\mathbb{E}}_{v\sim\mathcal{D}} \mathbf{I}_{\exists i,j, q_{ij}(v_{ij})\le\epsilon_1} Rev(\mathcal{M}_{BVCG}(v;\mathcal{I})) \le \frac{\epsilon}{10(1+\epsilon)} OPT(\mathcal{I}). \qquad (30)
\end{aligned}$$

The first inequality above is because, for each player i and valuation profile v, $\mathbf{I}_{\exists j, q_{ij}(v_{ij})\le\epsilon_1} \le \mathbf{I}_{\exists i,j, q_{ij}(v_{ij})\le\epsilon_1}$. The second inequality is by the Small-Tail Assumption 1.

Combining Equations 29 and 30, we have

$$Rev(\mathcal{M}_{BVCG}(\mathcal{I})) \le (1+\delta)^m Rev(\mathcal{M}_{EQBVCG}(\mathcal{I})) + \frac{\epsilon}{10(1+\epsilon)} OPT(\mathcal{I}).$$

By the construction of Mechanism 9, $(1+\delta)^m = 1+\frac{\epsilon}{5}$. Therefore Lemma 4 holds. □

Theorem 7 (restated). *$\forall\epsilon > 0$, any additive Bayesian instance $\mathcal{I} = (N, M, \mathcal{D})$ satisfying Small-Tail Assumption 1, $\mathcal{M}_{EQA}$ is DSIC, has query complexity $O(-m^2 n \log_{1+\frac{\epsilon}{5}} h(\frac{\epsilon}{10(1+\epsilon)}))$, and*

$$Rev(\mathcal{M}_{EQA}(\mathcal{I})) \ge \frac{1}{8(1+\epsilon)} OPT(\mathcal{I}).$$

Proof. First, as both $\mathcal{M}_{EQBVCG}$ and $\mathcal{M}_{EQIM}$ are DSIC, $\mathcal{M}_{EQA}$ is DSIC. Second, note that $\mathcal{M}_{EQA}$ runs both mechanisms with $\delta = (1+\frac{\epsilon}{5})^{1/m} - 1$ and $\epsilon_1 = h(\frac{\epsilon}{10(1+\epsilon)})$. To ease the analysis, when running mechanism $\mathcal{M}_{EQIM}$, let $\delta = \frac{\epsilon}{15}$ and $\epsilon_1 = h(\frac{2\epsilon}{3(5+\epsilon)})$: that is, set $\epsilon' = \frac{\epsilon}{5}$ and run mechanism $\mathcal{M}_{EQM}$ with parameter ϵ' for each item. By Theorem 5, with $O(-mn \log_{1+\frac{\epsilon}{15}} h(\frac{2\epsilon}{3(5+\epsilon)}))$ queries,

$$Rev(\mathcal{M}_{EQIM}(\mathcal{I})) \ge \frac{1}{1+\frac{\epsilon}{5}} Rev(\mathcal{M}_{IM}(\mathcal{I})).$$

By Lemma 4, with $O(-m^2 n \log_{1+\frac{\epsilon}{5}} h(\frac{\epsilon}{10(1+\epsilon)}))$ queries,

$$Rev(\mathcal{M}_{EQBVCG}(\mathcal{I})) \ge \frac{1}{1+\frac{\epsilon}{5}} \left(Rev(\mathcal{M}_{BVCG}(\mathcal{I})) - \frac{\epsilon}{10(1+\epsilon)} OPT(\mathcal{I})\right).$$

Note that the total query complexity is still $O(-m^2 n \log_{1+\frac{\epsilon}{5}} h(\frac{\epsilon}{10(1+\epsilon)}))$.

Let mechanism $\mathcal{M}_{EQA}$ run $\mathcal{M}_{EQBVCG}$ with probability $\frac{1}{4}$ and $\mathcal{M}_{EQIM}$ with probability $\frac{3}{4}$. We have

$$
\begin{aligned}
&Rev(\mathcal{M}_{EQA}(\mathcal{I})) = \frac{1}{4}Rev(\mathcal{M}_{EQBVCG}(\mathcal{I})) + \frac{3}{4}Rev(\mathcal{M}_{EQIM}(\mathcal{I})) \\
\geq\ & \frac{1}{4(1+\frac{\epsilon}{5})}\left(Rev(\mathcal{M}_{BVCG}(\mathcal{I})) - \frac{\epsilon}{10(1+\epsilon)}OPT(\mathcal{I})\right) + \frac{3}{4(1+\frac{\epsilon}{5})}Rev(\mathcal{M}_{IM}(\mathcal{I})) \\
\geq\ & \frac{1}{1+\frac{\epsilon}{5}}\left(\frac{1}{4}Rev(\mathcal{M}_{BVCG}(\mathcal{I})) + \frac{3}{4}Rev(\mathcal{M}_{IM}(\mathcal{I})) - \frac{\epsilon}{10(1+\epsilon)}OPT(\mathcal{I})\right) \\
\geq\ & \frac{1}{1+\frac{\epsilon}{5}}\left(\frac{1}{8}OPT(\mathcal{I}) - \frac{\epsilon}{10(1+\epsilon)}OPT(\mathcal{I})\right) = \frac{1}{8(1+\epsilon)}OPT(\mathcal{I}).
\end{aligned}
$$

The last inequality above holds because $2\mathcal{M}_{BVCG}(\mathcal{I}) + 6\mathcal{M}_{IM}(\mathcal{I}) \geq OPT(\mathcal{I})$ [5]. Thus Theorem 7 holds. □

C.3 The Proof for Corollary 1

Corollary 1 (restated)**.** *For any $\epsilon > 0$, $H > 1$, and prior distribution $\mathcal{D}$ with each $\mathcal{D}_{ij}$ bounded within $[1, H]$, there exist DSIC mechanisms that use $O(mn \log_{1+\epsilon} \frac{nmH(1+\epsilon)}{\epsilon})$ quantile queries for single-item auctions and unit-demand auctions, and use $O(m^2 n \log_{1+\epsilon} \frac{nmH(1+\epsilon)}{\epsilon})$ quantile queries for additive auctions, whose approximation ratios for OPT are respectively $1+\epsilon$, $24(1+\epsilon)$ and $8(1+\epsilon)$.*

Proof. We only need to show that the Small-Tail Assumptions 1 and 2 are naturally satisfied when the distributions have bounded supports. For example, consider additive auctions where all values are in $[1, H]$, as considered in [23, 11]. Then mH and 1 are straightforward upper- and lower-bounds for $OPT(\mathcal{I})$, respectively. Moreover, by individual rationality, mH is an upper-bound for the revenue generated under any valuation profiles. Given δ_1, let $\epsilon_1 = h(\delta_1) = \frac{\delta_1}{m^2 nH}$ and denote by E the event that there exist at least one player i and one item j with $q_{ij}(v_{ij}) \leq \epsilon_1$. By the union bound, $\Pr[E] \leq mn\epsilon_1 = mn \cdot \frac{\delta_1}{m^2 nH} = \frac{\delta_1}{mH}$. Therefore

$$
\mathop{\mathbb{E}}_{v\sim\mathcal{D}} \mathbf{I}_{\exists i,j,q_{ij}(v_{ij})\leq\epsilon_1} Rev(\mathcal{M}(v;\mathcal{I})) \leq mH \cdot \Pr[E] \leq \delta_1 \leq \delta_1 OPT(\mathcal{I}).
$$

Combining this observation with Theorems 5, 6 and 7, we have Corollary 1 when the values are all bounded in $[1, H]$. □

D Missing Proofs for Section 6

Theorem 8 (restated)**.** *$\forall \epsilon > 0$ and $\gamma \in (0, 1)$, for any Bayesian instance $\mathcal{I} = (N, M, \mathcal{D})$,*

- *for single-item auctions satisfying the Small-Tail Assumption 2, with $\tilde{O}(h^{-2}(\frac{2\epsilon}{3(1+\epsilon)}) \cdot (\frac{\epsilon}{1+\epsilon})^{-2})$ samples, mechanism $\mathcal{M}_{SM}$ achieves revenue at least $\frac{1}{1+\epsilon}OPT(\mathcal{I})$ with probability at least $1-\gamma$;*
- *for unit-demand auctions satisfying the Small-Tail Assumption 2, with $\tilde{O}(h^{-2}(\frac{2\epsilon}{3(1+\epsilon)}) \cdot (\frac{\epsilon}{1+\epsilon})^{-2})$ samples, mechanism $\mathcal{M}_{SM}$ achieves revenue at least $\frac{1}{24(1+\epsilon)}OPT$ with probability at least $1-\gamma$;*
- *for additive auctions satisfying the Small-Tail Assumption 1, with $\tilde{O}(h^{-2}(\frac{\epsilon}{10(1+\epsilon)})(\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}}))^{-2})$ samples, mechanism $\mathcal{M}_{SM}$ achieves revenue at least $\frac{1}{8(1+\epsilon)}OPT$ with probability at least $1-\gamma$.*

Proof. After constructing the distributions, we simply run the existing DSIC mechanisms as a Blackbox, and if the constructed distribution satisfies the property that for any quantile q_l,

$$q_{ij}(v_{ij}^{t\cdot q_{l+1}}) \geq \frac{1}{1+\frac{\epsilon}{3}} \left(q_{ij}(v_{ij}^{t\cdot q_l}) \right). \tag{31}$$

all our query complexity results for single-item and unit-demand auctions directly apply here.

Since here for sampling mechanism, we slice the quantile interval uniformly, in the ideal case, the selected sampled values correspond to the desired quantiles and $\mathcal{D}_{ij}(v_{ij}^{t\cdot q_l}) = \mathcal{D}_{ij}(v_{ij}^{t\cdot q_{l+1}})$. However, since these samples are random, we may not obtain the ideal case. In fact, given parameter $d = \frac{12+3\epsilon}{\epsilon}$, if for any quantile q_l,

$$q_l - \frac{q_l}{d} \leq q_{ij}(v_{ij}^{t\cdot q_l}) \leq q_l + \frac{q_l}{d}, \tag{32}$$

then

$$\frac{q_{ij}(v_{ij}^{t\cdot q_{l+1}})}{q_{ij}(v_{ij}^{t\cdot q_l})} \geq \frac{q_{l+1}(1-\frac{1}{d})}{q_l(1+\frac{1}{d})} \geq \frac{\frac{1}{1+\frac{\epsilon}{6}}(1-\frac{1}{d})}{1+\frac{1}{d}} = \frac{1}{1+\frac{\epsilon}{3}},$$

for any $\epsilon > 0$, that is, Equantion 31 holds. In the following, we show how many samples are enough to obtain Inequality 32.

First, we bound the probability that $v_{ij}^{t\cdot q_l}$ locates in the quantile interval $[q_l - \frac{q_l}{d}, q_l + \frac{q_l}{d}]$. Let $E_{ij,l}^{left}$ be the event that $v_{ij}^{t\cdot q_l}$ locates in the quantile interval $[0, q_l - \frac{q_l}{d}]$, and $E_{ij,l}^{right}$ be the event that $v_{ij}^{t\cdot q_l}$ locates in the quantile interval $[q_l + \frac{q_l}{d}, 1]$. Then

$$\Pr[E_{ij,l}^{left}] = \sum_{s=0}^{t-t\cdot q_l} \binom{t}{s} \left(q_l - \frac{q_l}{d}\right)^s \left(1 - q_l + \frac{q_l}{d}\right)^{t-s},$$

and

$$\Pr[E_{ij,l}^{right}] = \sum_{s=0}^{t\cdot q_l} \binom{t}{s} \left(1 - q_l - \frac{q_l}{d}\right)^s \left(q_l + \frac{q_l}{d}\right)^{t-s}.$$

By Chernoff's inequality and $\forall i, j, l$, letting $\Pr[E_{ij,l}^{left}]$ and $\Pr[E_{ij,l}^{right}]$ be no more than $\frac{\gamma}{2mn(k+1)}$, $t = \tilde{O}((\frac{\epsilon_1}{d})^{-2}) = \tilde{O}((\frac{\epsilon\cdot\epsilon_1}{(1+\epsilon)})^{-2})$. That is with $\tilde{O}(h^{-2}(\frac{2\epsilon}{3(1+\epsilon)})\cdot(\frac{\epsilon}{1+\epsilon})^{-2})$ samples, the probability that $v_{ij}^{t\cdot q_l}$ does not locate in the quantile interval $[q_l - \frac{q_l}{d}, q_l + \frac{q_l}{d}]$ is less than $\frac{\gamma}{mn(k+1)}$. By union bound, there exists one $v_{ij}^{t\cdot q_l}$ for all $i \in [n], j \in [m], l \in [k+1]$ does not locate in the quantile interval $[q_l - \frac{q_l}{d}, q_l + \frac{q_l}{d}]$ is less than γ. Then with probability $1-\gamma$, Inequality 32 holds.

For additive auctions, if the constructed distribution satisfies the property that for any quantile q_l,

$$q_{ij}(v_{ij}^{t\cdot q_l}) - q_{ij}(v_{ij}^{t\cdot q_{l+1}}) \geq \frac{1}{(1+\frac{\epsilon}{5})^{1/m}} \left(q_{ij}(v_{ij}^{t\cdot q_{l+1}}) - q_{ij}(v_{ij}^{t\cdot q_{l+2}}) \right), \tag{33}$$

all our query complexity results for additive auctions directly apply here. In fact, if for any quantile q_l,

$$q_l - \epsilon_1(\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}}) \leq q_{ij}(v_{ij}^{t\cdot q_l}) \leq q_l + \epsilon_1(\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}}),$$

then,

$$\frac{q_{ij}(v_{ij}^{t\cdot q_l}) - q_{ij}(v_{ij}^{t\cdot q_{l+1}})}{q_{ij}(v_{ij}^{t\cdot q_{l+q}}) - q_{ij}(v_{ij}^{t\cdot q_{l+2}})} \geq \frac{\epsilon_1 - \epsilon_1(1 - \frac{2}{1+(1+\frac{\epsilon}{5})^{1/m}})}{\epsilon_1 + \epsilon_1(1 - \frac{2}{1+(1+\frac{\epsilon}{5})^{1/m}})} = \frac{1}{(1+\frac{\epsilon}{5})^{1/m}},$$

Using the same technique of applying the Chernoff's inequality, with $\tilde{O}(h^{-2}(\frac{\epsilon}{10(1+\epsilon)})(\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}}))^{-2})$ samples, Equation 33 holds with probability $1-\gamma$. Thus Theorem 8 holds. □

E Missing Proofs for Section 7

E.1 Lower Bound

We only prove Theorem 9 for the single-player case, as in the following lemma. The lower bound for general multi-player single-item auctions can be proved using the same technique as in Theorem 1, thus the full proof has been omitted.

Lemma 5. *For any constant $\epsilon \in (0, \frac{1}{64})$, there exists a constant C such that, for any DSIC Bayesian mechanism $\mathcal{M}$ making less than C/ϵ non-adaptive value and quantile queries to the oracle, there exists a single-player single-item Bayesian auction instance $\mathcal{I} = (N, M, \mathcal{D})$ where $\mathcal{D}$ is regular, such that $Rev(\mathcal{M}(\mathcal{I})) < \frac{OPT(\mathcal{I})}{1+\epsilon}$.*

Proof. Since the distributions are unbounded, we can always construct the distributions such that for any finite number of value queries, the responses for the value queries have almost none contribution to the optimal revenue. Thus we only need to focus on the lower bound for quantile queries.

Letting $k \triangleq \lceil \frac{1}{\delta\epsilon} \rceil$ and $C \triangleq \frac{1-2\delta\epsilon}{2}$. Here δ is a constant to be determined later and δ, ϵ satisfies that $k \geq 2$. In our construction, we divide the quantile interval $[0, 1]$ into $k + 1$ sub-intervals each, with the right-end points defined as follows: from left to right, $q_0 = 0$, $q_{t+1} = q_t + \delta\epsilon$ for each $t \in \{0, \ldots, k-1\}$.

Accordingly, for any Bayesian mechanism $\mathcal{M}$ that makes less than $\frac{C}{\epsilon}$ non-adaptive quantile queries, there exists a quantile interval (q_t, q_{t+1}) such that, $q_{t+1} \leq 1 - 2\delta\epsilon$ and with probability at least $\frac{1}{2}$, no quantile in (q_t, q_{t+1}) is queried for $\mathcal{D}$ either. Indeed, if this is not the case, then with probability at least $\frac{1}{2}$, all the quantile intervals except $(1 - 2\delta\epsilon, 1 - \delta\epsilon)$ and $(1 - \delta\epsilon, 1)$ are queried. Since there are at least $k - 2$ quantile intervals, the expected total number of queries made by $\mathcal{M}$ is at least $\frac{k}{2} - 1 \geq \frac{1-2\delta\epsilon}{2\delta\epsilon} = \frac{C}{\epsilon}$, a contradiction.

We now construct two different single-player single-item Bayesian instances

$$\{\mathcal{I}_z = (N, M, \mathcal{D}_z)\}_{z \in \{1,2\}},$$

where the distributions outside the quantile range (q_t, q_{t+1}) are all the same. Thus with probability at least $\frac{1}{2}$, mechanism $\mathcal{M}$ cannot distinguish the $\mathcal{I}_z$'s from each other. We then show that when this happens, mechanism $\mathcal{M}$ cannot be a $(1 + 3\epsilon)$-approximation for all instances $\mathcal{I}_z$.

Let R be a parameter that are large enough such that no value query will get any useful response. Then the first distribution $\mathcal{D}_1$ with value bounded within $[0, \frac{R}{q_t}]$ is defined as follows, where $F_1(\cdot)$ is the cumulative probability function of $\mathcal{D}_1$.

$$F_1(v) = \begin{cases} 1 - \frac{R}{(1-q_{t+1})v+R}, & 0 \leq v < \frac{R}{q_t}, \\ 1, & v = \frac{R}{q_t}. \end{cases}$$

That is there is a probability mass $\frac{q_t}{1-\delta\epsilon}$ at value $\frac{R}{q_t}$ and within interval $[0, \frac{R}{q_t})$ it is a continuous distribution. Then for any quantile in range $(0, \frac{q_t}{1-\delta\epsilon}]$, the oracle will response $\frac{R}{q_t}$. For quantile q in range $(\frac{q_t}{1-\delta\epsilon}, 1]$, the oracle will response $v(q) = \frac{R}{1-q_{t+1}}(\frac{1}{q} - 1)$. Therefore the revenue function with related to the quantile q is

$$R_1(q) = \begin{cases} \frac{R}{1-q_{t+1}}(1-q), & \frac{q_t}{1-\delta\epsilon} < q \leq 1, \\ \frac{R}{1-\delta\epsilon}, & q = \frac{q_t}{1-\delta\epsilon}. \end{cases}$$

The revenue curve $R_1(q)$ is illustrated figure 2.

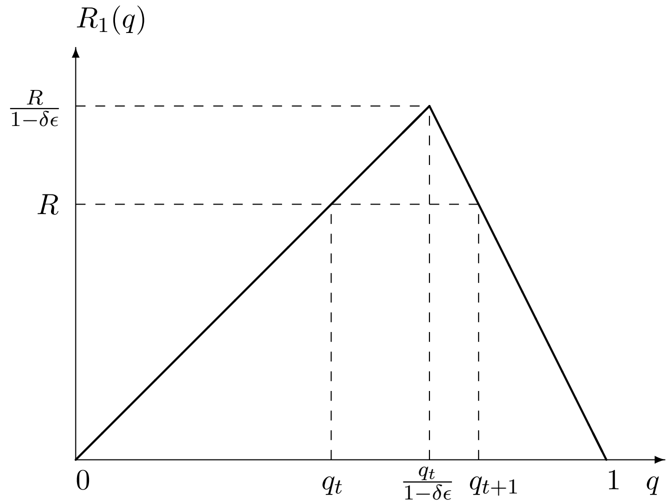

Figure 2: The revenue curve of $\mathcal{D}_1$.

The second distribution $\mathcal{D}_2$ with value bounded within $[0, \frac{R}{q_t}]$ is defined as follows, where $F_2(\cdot)$ is the cumulative probability function of $\mathcal{D}_2$. Let $v^* = \frac{R(2-\delta\epsilon)}{2(1-\delta\epsilon)-(2-\delta\epsilon)(1-q_{t+1})}$. Since $q_{t+1} \leq 1 - 2\delta\epsilon$, $v^* > 0$ is well defined and it is easy to check $v^* < \frac{R}{q_t}$.

$$F_2(v) = \begin{cases} 1 - \frac{R}{(1-q_{t+1})v+R}, & 0 \leq v < v^*, \\ 1 - \frac{R(1-\delta\epsilon)}{(1+q_t-\delta\epsilon)v-R}, & v^* \leq v < \frac{R}{q_t}, \\ 1, & v = \frac{R}{q_t}. \end{cases}$$

That is, there is a probability mass q_t at value $\frac{R}{q_t}$ and a two-step continuous distribution within $[q_t, q^*]$ and $[q^*, q_{t+1}]$. Thus for any quantile in range $(0, q_t]$, the oracle will response $\frac{R}{q_t}$. It can be calculated that the quantile of value v^* is $q^* = 1 - \frac{2-\delta\epsilon}{2(1-\delta\epsilon)} \cdot (1 - q_{t+1})$. Then for quantile q in range $(q_t, q^*]$, the oracle will response $v(q) = \frac{R}{q}(1 - \frac{q_t}{1+q_t-\delta\epsilon}) + \frac{R}{1+q_t-\delta\epsilon}$. For quantile q in range $(q^*, 1]$, the oracle will response $v(q) = \frac{R}{1-q_{t+1}}(\frac{1}{q} - 1)$. Therefore the revenue function with related to the quantile q is

$$R_2(q) = \begin{cases} \frac{R}{1-q_{t+1}}(1-q), & q^* < q \leq 1, \\ \frac{R}{1+q_t-\delta\epsilon}(1+q-\delta\epsilon), & q^* \leq q < q_t, \\ R, & q = q_t. \end{cases}$$

The revenue curve $R_2(q)$ is illustrated figure 3.

Indeed when the quantile query is from $[0, q_t] \cup [q_{t+1}, 1]$, the oracle's answers for all distributions are the same. Accordingly, with probability at least $\frac{1}{2}$, mechanism $\mathcal{M}$ cannot distinguish $\mathcal{D}_z$'s from each other, which means it cannot distinguish $\mathcal{I}_z$'s from each other, as desired.

Since $\mathcal{M}$ is truthful, the allocation rule for the player must be monotone and he will pay the threshold payment set by $\mathcal{M}$, denoted by P. Let $P^* = \frac{(4-\delta\epsilon)R}{4(1-\delta\epsilon)-(4-\delta\epsilon)(1-q_{t+1})}$. Here P may be randomized. Recall that $OPT(\mathcal{I}_1) = \frac{R}{1-\delta\epsilon}$ If with probability $\frac{1}{2}$ setting the price $P \leq P^*$, then for instance $\mathcal{I}_1$, we have

$$Rev(\mathcal{M}(\mathcal{I}_1)) \leq \frac{1}{2}OPT(\mathcal{I}_1) + \frac{1}{2}(\frac{3R}{4(1-\delta\epsilon)} + \frac{R}{4})$$
$$= \frac{7R}{8(1-\delta\epsilon)} + \frac{R}{8} = \frac{R}{1-\delta\epsilon}\left(1 - \frac{1}{8}\delta\epsilon\right) < \frac{OPT(\mathcal{I}_1)}{1+4\epsilon}$$

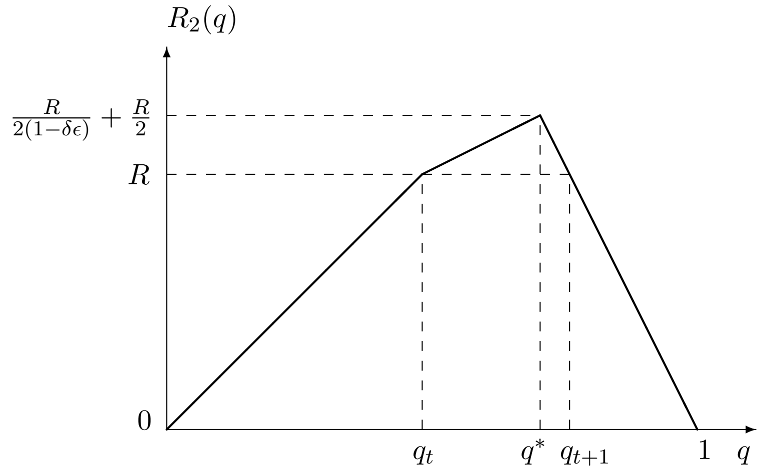

Figure 3: The revenue curve of $\mathcal{D}_2$.

when $\delta \geq 32$. On the other hand, recall that $OPT(\mathcal{I}_2) = \frac{R}{2(1-\delta\epsilon)} + \frac{R}{2} = \frac{(2-\delta\epsilon)R}{2(1-\delta\epsilon)}$. If with probability $\frac{1}{2}$, the price $P > P^*$, for instance $\mathcal{I}_2$, we have

$$Rev(\mathcal{M}(\mathcal{I}_2)) < \frac{1}{2}OPT(\mathcal{I}_2) + \frac{(4-\delta\epsilon)R}{2(4-2\delta\epsilon)} = \frac{(2-\delta\epsilon)R}{4(1-\delta\epsilon)} + \frac{(4-\delta\epsilon)R}{2(4-2\delta\epsilon)}$$
$$= \frac{(2-\delta\epsilon)R}{2(1-\delta\epsilon)}\left(\frac{1}{2} + \frac{(4-\delta\epsilon)(1-\delta\epsilon)}{2(2-\delta\epsilon)^2}\right) = OPT(\mathcal{I}_2)\left(1 - \frac{\delta\epsilon}{2(2-\delta\epsilon)^2}\right) < \frac{OPT(\mathcal{I}_2)}{1+4\epsilon}$$

when $\delta \geq 32$. Thus for any mechanism $\mathcal{M}$ with $O(\frac{1}{\epsilon})$ quantile queries, there exists $z^* \in \{0,1\}$ such that when $\epsilon < \frac{1}{64}$ and $\delta = 32$,

$$Rev(\mathcal{M}(\mathcal{I}_{z^*})) \leq \frac{OPT(\mathcal{I}_{z^*})}{2} + \frac{OPT(\mathcal{I}_{z^*})}{2(1+4\epsilon)} < \frac{OPT(\mathcal{I}_{z^*})}{1+\epsilon}.$$

Therefore Lemma 5 holds. □

E.2 Upper Bound

Mechanism 11 Efficient quantile Myerson mechanism for regular distributions, $\mathcal{M}_{EMR}$

1: Given $\epsilon > 0$, run algorithm $\mathcal{A}_Q$ with $\delta = \frac{\epsilon}{4}$ and $\epsilon_1 = \frac{\epsilon^2}{256n}$ for each player i's distribution $\mathcal{D}_i$, with the returned distribution denoted by $\mathcal{D}'_i$. Let $\mathcal{D}' = \times_{i\in N}\mathcal{D}'_i$.
2: Run $\mathcal{M}_{MRS}$ with $\mathcal{D}'$ and the players' reported values, $b = (b_i)_{i\in N}$, to get allocation $x = (x_i)_{i\in N}$ and price profile $p = (p_i)_{i\in N}$ as the outcome.

Theorem 10. (restated) $\forall \epsilon \in (0,1)$, *for any single-item instance* $\mathcal{I} = (N, M, \mathcal{D})$ *where* $\mathcal{D}$ *is regular, mechanism* $\mathcal{M}_{EMR}$ *is DSIC, has query complexity* $O(n \log_{1+\epsilon} \frac{n}{\epsilon})$, *and* $Rev(\mathcal{M}_{EMR}(\mathcal{I})) \geq \frac{OPT(\mathcal{I})}{1+\epsilon}$.

Proof. Consider the quantile value $q^* = \frac{\epsilon^2}{256n}$ and $v_i^* = F_i^{-1}(1-q^*)$. Let $\hat{v}_i = \max\{v_i^*, \frac{16OPT(\mathcal{I})}{\epsilon}\}$, and let $\bar{\mathcal{D}}_1, \ldots, \bar{\mathcal{D}}_n$ be imaginary distributions obtained by truncating $\mathcal{D}_1, \ldots, \mathcal{D}_n$ at $\hat{v}_i$, i.e., a sample $\bar{v}_i$ from $\bar{\mathcal{D}}_i$ is obtained by first sampling v_i from $\mathcal{D}_i$. and then letting $\bar{v}_i = \min\{v_i, \hat{v}_i\}$. Let $\bar{\mathcal{I}} = (N, M, \bar{\mathcal{D}})$.

Note that $\mathcal{D}'$ is also a discretization distribution for $\bar{\mathcal{D}}$, following the proof and notations of Theorem 5, letting v_i^- be the value first sampled from $\bar{\mathcal{D}}_i$ than rounding down to the support of $\mathcal{D}'$, we have $\mathcal{M}_{EMR}$ is truthful and using the technique of Mechanism 7, we have

$$\begin{aligned}
& Rev(\mathcal{M}_{EMR}(\mathcal{I})) = Rev(\mathcal{M}_{MRS}(v, \mathcal{D}')) \geq Rev(\mathcal{M}_{MRS}(v', \mathcal{D}')) \\
= & \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{\mathcal{D}}_{-i}} p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] \\
= & \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{\mathcal{D}}_{-i}} p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] \cdot (\mathbf{I}_{v_i^* \leq \frac{16OPT(\mathcal{I})}{\epsilon}} + \mathbf{I}_{v_i^* > \frac{16OPT(\mathcal{I})}{\epsilon}}).
\end{aligned} \tag{34}$$

We bound the indicators separately.

$$\begin{aligned}
& \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{\mathcal{D}}_{-i}} p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] \cdot \mathbf{I}_{v_i^* \leq \frac{16OPT(\mathcal{I})}{\epsilon}} \\
= & \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{\mathcal{D}}_{-i}} p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] \cdot \mathbf{I}_{v_i^* \leq \frac{16OPT(\mathcal{I})}{\epsilon}} \cdot (\mathbf{I}_{p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) < v_i^*} + \mathbf{I}_{p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \geq v_i^*}) \\
\geq & \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{\mathcal{D}}_{-i}} [p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \frac{1}{1+\frac{\epsilon}{4}} \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] \cdot \mathbf{I}_{v_i^* \leq \frac{16OPT(\mathcal{I})}{\epsilon}} \cdot \mathbf{I}_{p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) < v_i^*} \\
& + (p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] - \frac{16OPT(\mathcal{I})}{\epsilon} \cdot \frac{\epsilon^2}{256n}) \cdot \mathbf{I}_{v_i^* \leq \frac{16OPT(\mathcal{I})}{\epsilon}} \cdot \mathbf{I}_{p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \geq v_i^*}] \\
\geq & \frac{1}{1+\frac{\epsilon}{4}} \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{\mathcal{D}}_{-i}} p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] \cdot \mathbf{I}_{v_i^* \leq \frac{OPT(\mathcal{I})}{16\epsilon}} - \frac{\epsilon}{16} \cdot OPT(\mathcal{I}).
\end{aligned} \tag{35}$$

The first inequality here holds because for price $p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) < v_i^*$, we have

$$\Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] \geq \frac{1}{1+\frac{\epsilon}{4}} \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})]$$

due to the structure of the quantile queries for $\mathcal{D}'$. For price $p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \geq v_i^*$, when $v_i^* \leq \frac{16OPT(\mathcal{I})}{\epsilon}$, by the regularity of $\mathcal{D}_i$, the optimal reserve corresponds to the quantile from $(\frac{\epsilon^2}{256n}, 1]$. Thus we have

$$\begin{aligned}
& p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] \geq 0 \\
\geq & p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] - v_i^* \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[\bar{v}_i \geq v_i^*] \\
\geq & p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] - \frac{16OPT(\mathcal{I})}{\epsilon} \cdot \frac{\epsilon^2}{256n}
\end{aligned}$$

since the expected revenue is non-decreasing for quantile range $[0, \frac{\epsilon^2}{256n}]$. Thus Equation 35 holds. Then for the second indicator for Equation 34, we have

$$\begin{aligned}
& \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{\mathcal{D}}_{-i}} p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] \cdot \mathbf{I}_{v_i^* > \frac{16OPT(\mathcal{I})}{\epsilon}} \\
\geq & \frac{1}{1+\frac{\epsilon}{4}} \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{\mathcal{D}}_{-i}} p_i(\bar{v}_{-i}; \bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i \sim \bar{\mathcal{D}}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i}; \bar{\mathcal{D}})] \cdot \mathbf{I}_{v_i^* > \frac{16OPT(\mathcal{I})}{\epsilon}}
\end{aligned} \tag{36}$$

also by the construction the quantile queries for $\mathcal{D}'$. Combining Equation 34, 35 and 36, we have

$$\begin{aligned}
& Rev(\mathcal{M}_{EMR}(\mathcal{I})) \\
\geq\ & \frac{1}{1+\frac{\epsilon}{4}} \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i}\sim\bar{\mathcal{D}}_{-i}} p_i(\bar{v}_{-i};\bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i\sim\bar{\mathcal{D}}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i};\bar{\mathcal{D}})] \cdot \mathbf{I}_{v_i^* \leq \frac{16OPT(\mathcal{I})}{\epsilon}} - \frac{\epsilon}{16} \cdot OPT(\mathcal{I}) \\
& + \frac{1}{1+\frac{\epsilon}{4}} \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i}\sim\bar{\mathcal{D}}_{-i}} p_i(\bar{v}_{-i};\bar{\mathcal{D}}) \cdot \Pr_{\bar{v}_i\sim\bar{\mathcal{D}}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i};\bar{\mathcal{D}})] \cdot \mathbf{I}_{v_i^* > \frac{16OPT(\mathcal{I})}{\epsilon}} \\
=\ & \frac{1}{1+\frac{\epsilon}{4}} Rev(\mathcal{M}_{MRS}(\bar{v},\bar{\mathcal{D}})) - \frac{\epsilon}{16} \cdot OPT(\mathcal{I})
\end{aligned}$$

By Lemma 2 of [14], for $0 \leq \epsilon \leq 1$, $Rev(\mathcal{M}_{MRS}(\bar{v},\bar{\mathcal{D}})) = OPT(\bar{\mathcal{I}}) \geq (1-\frac{\epsilon}{4})OPT(\mathcal{I})$. Thus we have

$$\begin{aligned}
& Rev(\mathcal{M}_{EMR}(\mathcal{I})) \\
\geq\ & \frac{1}{1+\frac{\epsilon}{4}}(1-\frac{\epsilon}{4})OPT(\mathcal{I}) - \frac{\epsilon}{16} \cdot OPT(\mathcal{I}) \geq \frac{1}{1+\epsilon} OPT(\mathcal{I}).
\end{aligned}$$

Thus Theorem 10 holds. □

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